Differential Equations Homework

p. 5 1-9 odds only

p. 11 1, 2, 3, 5 #5 solve 3rd degree poly. eqn. by looking for rational zeros.

p. 21 1, 2, 3, 5, 6 #2 (a) $y = (x^2+2)e^{-x}$
(b) $y = (x^2+3)e^{-x}$

p. 36 1-17 odds only

p. 46 1, 5, 6, 7, 9, 11, 15, 18, 19

p. 56 1, 3, 5, 7, 9, 15, 17, 21

p. 67 1, 4, 5, 7, 13

p. 88 1, 3, 7, 11, 17

p. 102 1, 5, 7, 9, 23, 25 #1 answer wrong (should be 35%)

p. 123 7, 9

p. 132 1, 5, 11

p. 143 1, 5, 7, 9, 17, 19, 21, 27, 31, 37, 45

p. 159 1, 3, 19, 31, 35, 39, 51, 53, 55

p. 169 1, 3, 7, 13, 17

p. 197 1, 5

p. 208 1, 3, 4, 7 (skip b), 9

p. 217 1, 3, 5 (identify steady-state & transient terms)

p. 232 1, 3, 5, 6 (identify steady-state & transient terms)

Test 2

p. 434 1, 3, 5 (Part (a) Taylor Series method only)

p. 447 1, 7

p. 454 1, 7

p. 462 1

$$y'' - 5y' + 6y = 0$$

p.123 #7

(a) show e^{2x} and e^{3x} are linearly indep. solns. on the interval $-\infty < x < \infty$.

$$\left.\begin{aligned} f_1(x) &= e^{2x} \\ f_1'(x) &= 2e^{2x} \\ f_1''(x) &= 4e^{2x} \end{aligned}\right\} \qquad \begin{aligned} &[4e^{2x}] - 5[2e^{2x}] + 6[e^{2x}] = 0 \\ &4e^{2x} - 10e^{2x} + 6e^{2x} = 0 \quad \checkmark \end{aligned}$$

$$\left.\begin{aligned} f_2(x) &= e^{3x} \\ f_2'(x) &= 3e^{3x} \\ f_2''(x) &= 9e^{3x} \end{aligned}\right\} \qquad \begin{aligned} &[9e^{3x}] - 5[3e^{3x}] + 6[e^{3x}] = 0 \\ &9e^{3x} - 15e^{3x} + 6e^{3x} = 0 \quad \checkmark \end{aligned}$$

so both are solutions.

$$W(e^{2x}, e^{3x}) = \begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} = 3e^{5x} - 2e^{5x}$$

$$= e^{5x}$$

which is never zero

$\therefore$ the two solns. are linearly independent for all values of x.

(b) General soln. $y_c = c_1 e^{2x} + c_2 e^{3x}$, for arbitrary c_1, c_2

(c) $y(0) = 2 \longrightarrow 2 = c_1 + c_2$ (i) $\longrightarrow -2(i)$

$y'(0) = 3$

$y_c' = 2c_1 e^{2x} + 3c_2 e^{3x}$

$\longrightarrow 3 = 2c_1 + 3c_2$ (ii)

$$\begin{aligned} -4 &= -2c_1 - 2c_2 \\ 3 &= 2c_1 + 3c_2 \\ \hline -1 &= c_2 \end{aligned}$$

(i) $2 = c_1 + (-1)$

$c_1 = 3$

soln. $y = 3e^{2x} - e^{3x}$

This soln. is unique, + defined for all real #s since the constant functions $a_0(x) = 1$, $a_1(x) = -5$, and $a_2(x) = 6$ are all continuous for all real #s (Th. 4.1, p.112)

$y'' - 2y' + y = 0$

p.123 #8

(a) show e^x and xe^x are linearly indep. solns. on the int. $-\infty < x < +\infty$.

$$\left.\begin{aligned} f_1(x) &= e^x \\ f_1'(x) &= e^x \\ f_1''(x) &= e^x \end{aligned}\right\} \qquad e^x - 2e^x + e^x = 0 \checkmark$$

$$\left.\begin{aligned} f_2(x) &= xe^x \\ f_2'(x) &= xe^x + e^x \\ f_2''(x) &= (xe^x + e^x) + e^x \\ &= xe^x + 2e^x \end{aligned}\right\} \qquad [xe^x + 2e^x] - 2[xe^x + e^x] + [xe^x] = 0 \checkmark$$

so both f_1, f_2 are solns.

$$W(e^x, xe^x) = \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix} = e^x(xe^x + e^x) - e^x(xe^x)$$

$$= xe^{2x} + e^{2x} - xe^{2x}$$

$= e^{2x}$ which is never zero for any value of x.

Hence, the 2 solns. are linearly indep. on $-\infty < x < +\infty$

(b) $y_c = c_1 e^x + c_2 xe^x$, for arbitrary c_1, c_2

(c) $y(0) = 1 \longrightarrow 1 = c_1 + c_2(0) \longrightarrow c_1 = 1$

$y'(0) = 4$

$y_c' = c_1 e^x + c_2 xe^x + c_2 e^x$

$\longrightarrow 4 = c_1 + c_2(0) + c_2$

$4 = 1 + c_2$

$c_2 = 3$

$y = e^x + 3xe^x$

This soln. is unique, + defined for all real #'s since the constant functions $a_0(x) = 1$, $a_1(x) = -2$, and $a_2(x) = 1$ are all cont. for all real #'s (Th. 4.1, p.112)

$$x^2y'' - 2xy' + 2y = 0$$

p.123 #9 (a) show x and x^2 are linearly indep. solns. of this eqn. on the interval $0<x<\infty$

$f_1(x) = x$
$f_1'(x) = 1$
$f_1''(x) = 0$

$$x^2[0] - 2x[1] + 2[x] = 0 \quad ✓$$

$f_2(x) = x^2$
$f_2'(x) = 2x$
$f_2''(x) = 2$

$$x^2[2] - 2x[2x] + 2[x^2] = 0$$
$$2x^2 - 4x^2 + 2x^2 = 0 \quad ✓$$

So both f_1, f_2 are solns. for any x

$$W(x, x^2) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2$$

which is never zero on the interval $0<x<\infty$
(in fact, it is never zero on $-\infty<x<0$ either)

(b) $y_c = c_1 x + c_2 x^2$, for arbitrary c_1, c_2

(c) $y(1) = 3 \longrightarrow 3 = c_1 + c_2$ (i)
$y'(1) = 2$
$y_c' = c_1 + 2c_2 x$
$2 = c_1 + 2c_2$ (ii)

$$\begin{array}{rl} \text{(i)} & 3 = c_1 + c_2 \\ -\text{(ii)} & -2 = -c_1 - 2c_2 \\ \hline & 1 = -c_2 \end{array}$$

$c_2 = -1$

(i) $3 = c_1 + (-1)$
$c_1 = 4$

$$y = 4x - x^2$$

This soln. is unique, + defined for all #'s in the interval $0<x<\infty$, + also on the interval $-\infty<x<0$, since the functions $a_0(x) = x^2$, $a_1(x) = -2x$, $a_2(x) = 2$ are all cont. on these ints. (Th. 4.1, p.112)

$y = x$ is a soln. of

(1) $x^2 y'' - 4xy' + 4y = 0.$

p.132 #1

Let $y = xv$

$y' = xv' + v$

$y'' = xv'' + 2v'$

Then (1) becomes $x^2(xv'' + 2v') - 4x(xv' + v) + 4(xv) = 0$

$x^3 v'' + (-2x^2) v' = 0$

Let $w = v'$

$x^3 \frac{dw}{dx} + (-2x^2) w = 0$

<u>separable</u> (also 1st-order homogeneous linear)

$x^3 \frac{dw}{dx} = 2x^2 w$

$\frac{dw}{w} = 2 \frac{dx}{x}$

integrating: $\ln w = 2 \ln x + \ln C$

$\ln w = \ln(Cx^2)$

$w = Cx^2 \longrightarrow w = x^2$ (choose $c=1$)

But $w = v'$, so $\frac{dv}{dx} = x^2$

$dv = x^2 dx$

$v = \frac{1}{3}x^3 + C_0$

$\longrightarrow v = \frac{1}{3}x^3$ (choose $C_0 = 0$)

so $g(x) = f(x)v(x) = x(\frac{1}{3}x^3) = \frac{1}{3}x^4$ is a linearly indep. soln. of (1).

So $y = c_1 x + c_0(\frac{1}{3}x^4)$

$= c_1 x + c_2 x^4$ is the general soln.

$y = e^{2x}$ is a soln. of

p. 132 # 5

(1) $(2x+1)y'' - 4(x+1)y' + 4y = 0$

Let $y = e^{2x} v$

$y' = e^{2x} v' + 2e^{2x} v$

$y'' = e^{2x} v'' + 2e^{2x} v' + 2e^{2x} v' + 4e^{2x} v$

$= e^{2x} v'' + 4e^{2x} v' + 4e^{2x} v$

substituting these expressions for y, y', y'' into (1)

$(2x+1)(e^{2x} v'' + 4e^{2x} v' + 4e^{2x} v) - 4(x+1)(e^{2x} v' + 2e^{2x} v) + 4(e^{2x} v) = 0$

$(2xe^{2x} + e^{2x}) v'' + (4xe^{2x}) v' = 0$

Let $w = v'$ $\quad (2xe^{2x} + e^{2x}) \frac{dw}{dx} + (4xe^{2x}) w = 0$

separable $\quad \frac{dw}{w} = \frac{-4xe^{2x}}{2xe^{2x} + e^{2x}} dx$

$\frac{dw}{w} = \frac{-4x\,dx}{2x+1} = \left(-2 + \frac{2}{2x+1}\right) dx$

integrating: $\quad \ln w = -2x + \ln(2x+1)$

$w = e^{-2x + \ln(2x+1)}$

$w = e^{-2x} e^{\ln(2x+1)}$

$w = e^{-2x}(2x+1)$

But $w = v'$ $\quad \frac{dv}{dx} = e^{-2x}(2x+1)$

$dv = (2xe^{-2x} + e^{-2x})\, dx$

integrating: $\quad v = \int 2xe^{-2x} dx - \frac{1}{2} e^{-2x}$

By parts $\quad u_0 = 2x \quad dv_0 = e^{-2x} dx$

$du_0 = 2\,dx \quad v_0 = -\frac{1}{2} e^{-2x}$

$v = (2x)(-\frac{1}{2} e^{-2x}) - \int (-\frac{1}{2} e^{-2x})(2\,dx) - \frac{1}{2} e^{-2x}$

(#5, p. 132 cont.)

$$v = -xe^{-2x} + \int e^{-2x}\,dx - \tfrac{1}{2} e^{-2x}$$

$$v = -xe^{-2x} - \tfrac{1}{2} e^{-2x} - \tfrac{1}{2} e^{-2x}$$

$$v = -xe^{-2x} - e^{-2x}$$

So $g(x) = f(x)\,v(x) = e^{2x}\left(-xe^{-2x} - e^{-2x}\right)$

$= -x - 1$ is a linearly independent soln. of (1).

The general soln. is

$$y = c_1 e^{2x} + c_0(-x-1)$$

$$= c_1 e^{2x} + c_2(x+1)$$

$y = 1/6$ is a particular soln. of

$$y'' - 5y' + 6y = 1$$

p. 133
#11

$y = x/6 + 5/36$ is a particular soln. of

$$y'' - 5y' + 6y = x$$

$y = \frac{e^x}{2}$ is a particular soln. of

$$y'' - 5y' + 6y = e^x$$

Then a particular soln. of

$$y'' - 5y' + 6y = 2 - 12x + 6e^x$$

is $y_p = 2(1/6) + (-12)(x/6 + 5/36) + 6(e^x/2)$

$$= 1/3 - 2x - 5/3 + 3e^x$$

$$= -4/3 - 2x + 3e^x$$

(see Th. 4.10, p. 131)

p.143 #1

$$y'' - 5y' + 6y = 0$$

the characteristic Eqn. is

$$m^2 - 5m + 6 = 0$$

$$(m-3)(m-2) = 0$$

$$m = 3 \qquad m = 2$$

General soln. $y = c_1 e^{3x} + c_2 e^{2x}$

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p.143 #5

$$2y'' + y' - 6y = 0$$

the characteristic eqn. is

$$2m^2 + m - 6 = 0$$

$$(2m - 3)(m + 2) = 0$$

$$m = 3/2 \qquad m = -2$$

General soln. $y = c_1 e^{3/2 x} + c_2 e^{-2x}$

p.143 #7

$$4y'' - 4y' + y = 0$$

characteristic Eqn. :

$$4m^2 - 4m + 1 = 0$$

$$(2m-1)(2m-1) = 0$$

$$m = 1/2 \qquad m = 1/2$$

Repeated Real Roots

General Soln. $y = c_1 e^{1/2 x} + c_2 x e^{1/2 x}$

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p.143 #9

$$y'' + 6y' + 11y = 0$$

characteristic Eqn. :

$$m^2 + 6m + 11 = 0$$

Does not factor $\rightarrow$ use quadratic formula

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-6 \pm \sqrt{(6)^2 - 4(1)(11)}}{2(1)}$$

$$= \frac{-6 \pm \sqrt{36 - 44}}{2}$$

$$= \frac{-6 \pm \sqrt{-8}}{2} = \frac{-6 \pm 2\sqrt{2}\, i}{2} = -3 \pm \sqrt{2}\, i$$

Complex Roots $a \pm bi = -3 \pm \sqrt{2}\, i$

i.e. $a = -3$
$b = \sqrt{2}$

General Soln. $y = e^{-3x}(c_1 \sin \sqrt{2}\, x + c_2 \cos \sqrt{2}\, x)$

(use formula $y = e^{ax}(c_1 \sin bx + c_2 \cos bx)$)

$$y'' - 8y' + 16y = 0$$

p.143 #17

Characteristic Eqn. :

$$m^2 - 8m + 16 = 0$$

$$(m-4)^2 = 0$$

$$m = 4 \qquad m = 4$$

General soln. $y = c_1 e^{4x} + c_2 x e^{4x}$

$$y'' - 4y' + 13y = 0$$

p.143 #19

Characteristic Eqn.

$$m^2 - 4m + 13 = 0$$

Does not factor

$$m = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{16 - 52}}{2}$$

$$= \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i$$

General soln. $y = e^{2x}(c_1 \sin 3x + c_2 \cos 3x)$

$$y'' + 9y = 0$$

characteristic Eqn.

p.143
#21

$$m^2 + 9 = 0$$

$$m^2 = -9$$

$$m = \pm\sqrt{-9} = \pm 3i$$

General Soln. $y = e^{0x}(c_1 \sin 3x + c_2 \cos 3x)$

$e^0 = 1$ $\quad y = c_1 \sin 3x + c_2 \cos 3x$

---

p.143
#27

$$y^{iv} + 8y'' + 16y = 0$$

characteristic Eqn.

$$m^4 + 8m^2 + 16 = 0$$

$$(m^2 + 4)^2 = 0$$

$m^2 + 4 = 0$ $\qquad m^2 + 4 = 0 \rightarrow m = \pm 2i$

$$m^2 = -4$$

$$m = \pm\sqrt{-4} = \pm 2i$$

Repeated complex Roots

General Soln. $y = e^{0x}(c_1 \sin 2x + c_2 x \sin 2x + c_3 \cos 2x + c_4 x \cos 2x)$

$$y = c_1 \sin 2x + c_2 x \sin 2x + c_3 \cos 2x + c_4 x \cos 2x$$

p.143
#31

$$y^{v} - 2y^{iv} + y''' = 0$$

characteristic Eqn.

$$m^5 - 2m^4 + m^3 = 0$$

$$m^3(m^2 - 2m + 1) = 0$$

$m^3 = 0$ $\quad$ $m^2 - 2m + 1 = 0$

$m=0 \quad m=0 \quad m=0$ $\quad$ $(m-1)(m-1) = 0$

$m=1 \quad m=1$

General Soln.

$$y = c_1 e^{0x} + c_2 x e^{0x} + c_3 x^2 e^{0x} + c_4 e^{1x} + c_5 x e^{1x}$$

$$y = c_1 + c_2 x + c_3 x^2 + c_4 e^{x} + c_5 x e^{x}$$

---

p.144
#37

$$y'' - y' - 12y = 0 \quad , \quad y(0) = 3 \quad y'(0) = 5$$

Characteristic Eqn.

$m^2 - m - 12 = 0$

$(m-4)(m+3) = 0$

$m = 4 \quad m = -3$

General Soln. $\quad y = c_1 e^{4x} + c_2 e^{-3x}$

$y(0) = 3 \rightarrow \quad 3 = c_1 + c_2 \quad (i)$

to use $y'(0) = 5$, we first need y':

$$y' = 4c_1 e^{4x} - 3c_2 e^{-3x}$$

$y'(0) = 5 \quad 5 = 4c_1 - 3c_2 \quad (ii)$

solving (i) + (ii) simultaneously:

$3(i) \quad 9 = 3c_1 + 3c_2$

$(ii) \quad 5 = 4c_1 - 3c_2$

$14 = 7c_1$

$c_1 = 2 \rightarrow (i)\ 3 = 2 + c_2$

$\rightarrow c_2 = 1$

$y = 2e^{4x} + e^{-3x}$

$$y'' - 4y' + 29y = 0 \quad , y(0) = 0 \quad , y'(0) = 5$$

p. 144
#45

characteristic Eqn.

$$m^2 - 4m + 29 = 0$$

$$m = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(29)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{16 - 116}}{2} = \frac{4 \pm \sqrt{-100}}{2} = \frac{4 \pm 10i}{2} = 2 \pm 5i$$

General Soln. $\quad y = e^{2x}(c_1 \sin 5x + c_2 \cos 5x)$

$y(0) = 0 \rightarrow \quad 0 = e^0 (c_1 \sin 0 + c_2 \cos 0)$

$$0 = 1(0 + c_2) \rightarrow c_2 = 0$$

$$y = e^{2x}(c_1 \sin 5x)$$

$$y' = e^{2x}(5c_1 \cos 5x) + 2e^{2x}(c_1 \sin 5x)$$

$y'(0) = 5 \rightarrow \quad 5 = e^0 (5c_1 \cos 0) + 2e^0 (c_1 \sin 0)$

$$5 = 1(5c_1) + 2(1)(0)$$

$$c_1 = 1$$

Hence $\quad y = e^{2x}(1 \cdot \sin 5x) = e^{2x} \sin 5x$

#59

$$y = C_1 e^{4x} + C_2 x e^{4x} + C_3 x^2 e^{4x} + C_4 x^3 e^{4x} +$$

$$\underbrace{e^{2x}\left[(C_5 + C_6 x + C_7 x^2)\sin 3x + (C_8 + C_9 x + C_{10} x^2)\cos 3x\right]}_{\text{Th. 4.13}}$$

$$y'' - 3y' + 2y = 4x^2$$

p.159 #1

Consider the corresponding homog. eqn.

$$y'' - 3y' + 2y = 0$$

The characteristic eqn. is

$$m^2 - 3m + 2 = 0$$

$$(m-2)(m-1) = 0$$

$$m = 2 \qquad m = 1$$

So the complementary soln. is

$$y_c = c_1 e^{2x} + c_2 e^{x}$$

The non homogeneous term $4x^2$ has a UC set

$$S_1 = \{x^2, x, 1\}$$

Since this set contains no solns. of the homog. eqn.,

$$y_p = Ax^2 + Bx + C$$

To find A, B, C we substitute into the original eqn.

$$y_p = Ax^2 + Bx + C$$

$$y_p' = 2Ax + B$$

$$y_p'' = 2A$$

$$2A - 3(2Ax + B) + 2(Ax^2 + Bx + C) = 4x^2$$

$$(2A)x^2 + (-6A + 2B)x + (2A - 3B + 2C) = 4x^2$$

matching coefficients of the x^2, x, and constant terms:

$$2A = 4 \longrightarrow A = 2$$

$$-6A + 2B = 0 \longrightarrow -6(2) + 2B = 0 \longrightarrow B = 6$$

$$2A - 3B + 2C = 0 \longrightarrow 2(2) - 3(6) + 2C = 0 \longrightarrow C = 7$$

$$y_p = 2x^2 + 6x + 7$$

So the general soln. is

$$y = y_c + y_p = c_1 e^{2x} + c_2 e^{x} + 2x^2 + 6x + 7$$

$$y'' + 2y' + 5y = 6\sin 2x + 7\cos 2x$$

p.159 #3

consider the corresponding homog. eqn.

$$y'' + 2y' + 5y = 0$$

characteristic eqn.

$$m^2 + 2m + 5 = 0$$

$$m = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(5)}}{2(1)} = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

$$y_c = e^{-x}(c_1 \sin 2x + c_2 \cos 2x)$$

$$= c_1 e^{-x} \sin 2x + c_2 e^{-x} \cos 2x$$

The nonhomogeneous function $6\sin 2x + 7\cos 2x$ has UC sets

$$S_1 = \{\sin 2x, \cos 2x\}$$

$$S_2 = \{\sin 2x, \cos 2x\}$$

since S_2 is merely a repeat of S_1, we consider only

$$S_1 = \{\sin 2x, \cos 2x\}$$

since this set contains no solns. of the homogeneous eqn.

$$y_p = A \sin 2x + B \cos 2x$$

$$y_p' = 2A \cos 2x - 2B \sin 2x$$

$$y_p'' = -4A \sin 2x - 4B \cos 2x$$

$$(-4A \sin 2x - 4B \cos 2x) + 2(2A \cos 2x - 2B \sin 2x) + 5(A \sin 2x + B \cos 2x) = 6\sin 2x + 7\cos 2x$$

$$(A - 4B)\sin 2x + (4A + B)\cos 2x = 6\sin 2x + 7\cos 2x$$

matching coefficients: $\left.\begin{array}{l} A - 4B = 6 \\ 4A + B = 7 \end{array}\right\} \to$ $\begin{array}{l} A - 4B = 6 \\ 16A + 4B = 28 \\ \hline 17A = 34 \end{array}$ $\to A = 2 \to B = -1$

$$y_p = 2\sin 2x - \cos 2x$$

So the general soln. is

$$y = y_c + y_p = c_1 e^{-x} \sin 2x + c_2 e^{-x} \cos 2x + 2\sin 2x - \cos 2x$$

$$y'' + y' - 6y = 10e^{2x} - 18e^{3x} - 6x - 11$$

p.160
#19

The corresponding homog. eqn.

$y'' + y' - 6y = 0$ has charact. eqn.

$$m^2 + m - 6 = 0$$
$$(m+3)(m-2) = 0$$
$$m = -3 \qquad m = 2$$
$$y_c = c_1 e^{-3x} + c_2 e^{2x}$$

The nonhomogeneous function $10e^{2x} - 18e^{3x} - 6x - 11$ has UC sets

$$S_1 = \{e^{2x}\}$$
$$S_2 = \{e^{3x}\}$$
$$S_3 = \{x, 1\}$$
$$S_4 = \{1\}$$

Since S_4 is contained in S_3, we'll omit S_4.

we must revise S_1 since e^{2x} is a solution of the homog. eqn.

$$S_1' = \{xe^{2x}\}$$

$$y_p = Axe^{2x} + Be^{3x} + Cx + D$$
$$y_p' = 2Axe^{2x} + Ae^{2x} + 3Be^{3x} + C$$
$$y_p'' = 4Axe^{2x} + \underbrace{2Ae^{2x} + 2Ae^{2x}}_{4Ae^{2x}} + 9Be^{3x}$$

$$(4Axe^{2x} + 4Ae^{2x} + 9Be^{3x}) + (2Axe^{2x} + Ae^{2x} + 3Be^{3x} + C) - 6(Axe^{2x} + Be^{3x} + Cx + D)$$
$$= 10e^{2x} - 18e^{3x} - 6x - 11$$

$$(4A + 2A - 6A)xe^{2x} + (4A + A)e^{2x} + (9B + 3B - 6B)e^{3x} + (-6C)x + (C - 6D)$$
$$= 10e^{2x} - 18e^{3x} - 6x - 11$$

$$5A = 10 \rightarrow A = 2$$
$$6B = -18 \rightarrow B = -3$$
$$-6C = -6 \rightarrow C = 1$$
$$C - 6D = -11 \rightarrow D = 2$$

$$y_p = 2xe^{2x} - 3e^{3x} + x + 2$$

$$y = y_c + y_p = c_1 e^{-3x} + c_2 e^{2x} + 2xe^{2x} - 3e^{3x} + x + 2$$

$$y'' + y = x \sin x$$

The corresponding homog. eqn.

#31
p.160

$y'' + y = 0$ has characteristic eqn.

$$m^2 + 1 = 0$$
$$m^2 = -1$$
$$m = \pm\sqrt{-1} = \pm i$$

$$y_c = e^{0x}(c_1 \sin x + c_2 \cos x) = c_1 \sin x + c_2 \cos x$$

The nonlinear term $x \sin x$ has UC set

$$S_1 = \{ x \sin x, x \cos x, \sin x, \cos x \}$$

But since $\sin x$, $\cos x$ are both solns. of the homog. eqn., we must mult. <u>every</u> elt. of S_1 by x

$$S_1' = \{ x^2 \sin x, x^2 \cos x, x \sin x, x \cos x \}$$

$$y_p = Ax^2 \sin x + Bx^2 \cos x + Cx \sin x + Dx \cos x$$

$$y_p' = Ax^2 \cos x + 2Ax \sin x - Bx^2 \sin x + 2Bx \cos x + Cx \cos x + C \sin x - Dx \sin x + D \cos x$$

$$y_p'' = -Ax^2 \sin x + 2Ax \cos x + 2Ax \cos x + 2A \sin x - Bx^2 \cos x - 2Bx \sin x - 2Bx \sin x + 2B \cos x - Cx \sin x + C \cos x + C \cos x - Dx \cos x - D \sin x - D \sin x$$

$$= -Ax^2 \sin x + 4Ax \cos x + 2A \sin x - Bx^2 \cos x - 4Bx \sin x + 2B \cos x - Cx \sin x + 2C \cos x - Dx \cos x - 2D \sin x$$

our original eqn. becomes:

$$\left(-Ax^2 \sin x + 4Ax \cos x + 2A \sin x - Bx^2 \cos x - 4Bx \sin x + 2B \cos x - Cx \sin x + 2C \cos x - Dx \cos x - 2D \sin x\right)$$
$$+ \left(Ax^2 \sin x + Bx^2 \cos x + Cx \sin x + Dx \cos x\right) = x \sin x$$

(#31 p. 160 cont.)

$$(-A \quad +A \quad)x^2 \sin x + (4A - D + D \quad)x\cos x + (2A - 2D)\sin x$$

$$+ (-B \quad + B \quad)x^2\cos x + (-4B - C + C \quad)x \sin x + (2B + 2C \quad)\cos x$$

$$= x \sin x$$

$4A = 0 \quad \rightarrow \quad A = 0$

$2A - 2D = 0. \quad \rightarrow \quad D = 0$

$-4B = 1 \quad \rightarrow \quad B = -1/4$

$2B + 2C = 0 \quad \rightarrow \quad C = 1/4$

$$y_p = -1/4\, x^2 \cos x + 1/4\, x \sin x$$

$$y = y_c + y_p = C_1 \sin x + C_2 \cos x - 1/4\, x^2 \cos x + 1/4\, x \sin x$$

$$y'' - 4y' + 3y = 9x^2 + 4, \quad y(0) = 6, \quad y'(0) = 8$$

p.161
#35

The homog. eqn.

$$y'' - 4y' + 3y = 0$$ has charact. eqn.

$$m^2 - 4m + 3 = 0$$
$$(m-3)(m-1) = 0$$
$$m = 3 \qquad m = 1$$

$$y_c = c_1 e^{3x} + c_2 e^{x}$$

The nonhomog. function $9x^2 + 4$ has UC sets

$$S_1 = \{x^2, x, 1\}$$
$$S_2 = \{1\}$$

Since S_2 is contained in S_1, we eliminate S_2.

$$y_p = Ax^2 + Bx + C$$
$$y_p' = 2Ax + B$$
$$y_p'' = 2A$$

$$(2A) - 4(2Ax + B) + 3(Ax^2 + Bx + c) = 9x^2 + 4$$

$$(3A)x^2 + (-8A + 3B)x + (2A - 4B + 3C) = 9x^2 + 4$$

$$3A = 9 \rightarrow A = 3$$
$$-8A + 3B = 0 \rightarrow B = 8$$
$$2A - 4B + 3C = 4 \rightarrow C = 10$$

$$y_p = 3x^2 + 8x + 10$$

$$y = y_c + y_p = c_1 e^{3x} + c_2 e^{x} + 3x^2 + 8x + 10$$

$y(0) = 6 \rightarrow 6 = c_1 + c_2 + 10 \rightarrow c_1 + c_2 = -4$ (i)

$$y' = 3c_1 e^{3x} + c_2 e^{x} + 6x + 8$$

$y'(0) = 8 \rightarrow 8 = 3c_1 + c_2 + 8 \rightarrow 3c_1 + c_2 = 0$ (ii)

solving (i) + (ii) simultaneously

$$\begin{array}{r} c_1 + c_2 = -4 \\ -3c_1 - c_2 = 0 \\ \hline -2c_1 = -4 \end{array} \rightarrow c_1 = 2 \rightarrow c_2 = -6$$

$$y = 2e^{3x} - 6e^{x} + 3x^2 + 8x + 10$$

$$y'' + 8y' + 16y = 8e^{-2x}, \quad y(0) = 2, \quad y'(0) = 0$$

p.161 #39

The corresponding homog. eqn.

$y'' + 8y' + 16y = 0$ has charact. eqn.

$$m^2 + 8m + 16 = 0$$

$$(m+4)(m+4) = 0$$

$$m = -4 \qquad m = -4$$

$$y_c = c_1 e^{-4x} + c_2 x e^{-4x}$$

The nonhomog. term $8e^{-2x}$ has UC set

$$S_1 = \{e^{-2x}\}$$

$$y_p = Ae^{-2x}$$

$$y_p' = -2Ae^{-2x}$$

$$y_p'' = 4Ae^{-2x}$$

$$(4Ae^{-2x}) + 8(-2Ae^{-2x}) + 16(Ae^{-2x}) = 8e^{-2x}$$

$$4Ae^{-2x} = 8e^{-2x}$$

$$A = 2$$

$$y_p = 2e^{-2x}$$

$$y = y_c + y_p = c_1 e^{-4x} + c_2 x e^{-4x} + 2e^{-2x}$$

$y(0) = 2 \longrightarrow \quad 2 = c_1 + 2 \quad \longrightarrow c_1 = 0$

$$y = c_2 x e^{-4x} + 2e^{-2x}$$

$$y' = -4c_2 x e^{-4x} + c_2 e^{-4x} - 4e^{-2x}$$

$y'(0) = 0 \qquad 0 = c_2 - 4 \qquad \longrightarrow c_2 = 4$

$$y = 4xe^{-4x} + 2e^{-2x}$$

Set-up only

p.161
#51

$$y'' - 6y' + 8y = x^3 + x + e^{-2x}$$

The corresponding homog. eqn.

$y'' - 6y' + 8y = 0$ has charact. eqn.

$m^2 - 6m + 8 = 0$

$(m-4)(m-2) = 0$

$m = 4 \quad m = 2$

$y_c = c_1 e^{4x} + c_2 e^{2x}$

The nonhomogeneous function $x^3 + x + e^{-2x}$ has UC sets

$S_1 = \{x^3, x^2, x, 1\}$

$S_2 = \{x, 1\}$

$S_3 = \{e^{-2x}\}$

since the set S_2 is completely contained in S_1, we'll eliminate S_2. None of the elts. of S_1 or S_3 are sol'ns of the homog. eqn., so

$y_p = Ax^3 + Bx^2 + Cx + D + Ee^{-2x}$

Set-up only

$$y'' + 4y' + 5y = e^{-2x}(1+\cos x)$$

p.161
\# 53

The corresponding homog. eqn.

$y'' + 4y' + 5y = 0$ has charact. eqn.

$$m^2 + 4m + 5 = 0$$

$$m = \frac{-4 \pm \sqrt{4^2 - 4(1)(5)}}{2(1)}$$

$$= \frac{-4 \pm \sqrt{-4}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i$$

$$y_c = e^{-2x}(c_1 \sin x + c_2 \cos x)$$

$$y_c = c_1 e^{-2x} \sin x + c_2 e^{-2x} \cos x$$

The nonhomog. function $e^{-2x}(1+\cos x)$ has UC sets

$$S_1 = \{e^{-2x}\}$$

$$S_2 = \{e^{-2x}\cos x, \ e^{-2x}\sin x\}$$

since the elts. of S_2 are solns. of the corresponding homog. eqn., we need

$$S_2' = \{xe^{-2x}\cos x, \ xe^{-2x}\sin x\}$$

$$y_p = Ae^{-2x} + Bxe^{-2x}\cos x + Cxe^{-2x}\sin x$$

Set-up only

$$y'' + 6y' + 13y = xe^{-3x}\sin 2x + x^2e^{-2x}\sin 3x$$

p.161 #55

The corresponding homog. eqn.

$y'' + 6y' + 13y = 0$ has charact. eqn.

$$m^2 + 6m + 13 = 0$$

$$m = \frac{-6 \pm \sqrt{6^2 - 4(1)(13)}}{2(1)} = \frac{-6 \pm \sqrt{-16}}{2} = \frac{-6 \pm 4i}{2} = -3 \pm 2i$$

$$y_c = e^{-3x}(c_1\sin 2x + c_2\cos 2x)$$

The nonhomog. function $xe^{-3x}\sin 2x + x^2e^{-2x}\sin 3x$

has UC sets

$$S_1 = \{xe^{-3x}\sin 2x,\ xe^{-3x}\cos 2x,\ e^{-3x}\sin 2x,\ e^{-3x}\cos 2x\}$$

$$S_2 = \{x^2e^{-2x}\sin 3x,\ x^2e^{-2x}\cos 3x,\ xe^{-2x}\sin 3x,\ xe^{-2x}\cos 3x,\ e^{-2x}\sin 3x,\ e^{-2x}\cos 3x\}$$

Since some of the elts. of S_1 are solns. of the homog. eqn., mult. S_1 by x:

$$S_1' = \{x^2e^{-3x}\sin 2x,\ x^2e^{-3x}\cos 2x,\ xe^{-3x}\sin 2x,\ xe^{-3x}\cos 2x\}$$

Using S_1' and S_2

$$y_p = Ax^2e^{-3x}\sin 2x + Bx^2e^{-3x}\cos 2x + Cxe^{-3x}\sin 2x + Dxe^{-3x}\cos 2x$$
$$+ Ex^2e^{-2x}\sin 3x + Fx^2e^{-2x}\cos 3x + Gxe^{-2x}\sin 3x + Hxe^{-2x}\cos 3x$$
$$+ Ie^{-2x}\sin 3x + Je^{-2x}\cos 3x$$

$$y'' + y = \cot x$$

The corresponding homog eqn.

p.169 #1

$$y'' + y = 0 \qquad \text{has charact. eqn.}$$

$$m^2 + 1 = 0$$

$$m = \pm i$$

So the complementary soln. is

$$y_c = e^{0x}(c_1 \sin x + c_2 \cos x)$$

$$y_c = c_1 \sin x + c_2 \cos x$$

we look for a particular soln. of the form

$$y_p = v_1(x) \sin x + v_2(x) \cos x$$

Then $y_p' = v_1(x) \cos x - v_2(x) \sin x + (v_1'(x) \sin x + v_2'(x) \cos x)$

we impose the requirement $(v_1'(x) \sin x + v_2'(x) \cos x) = 0$

Hence $y_p' = v_1(x) \cos x - v_2(x) \sin x$

Then $y_p'' = v_1'(x) \cos x - v_1(x) \sin x - v_2'(x) \sin x - v_2(x) \cos x$

substituting y_p, y_p'' into our original eqn.

$$v_1'(x)\cos x - v_1(x)\sin x - v_2'(x)\sin x - v_2(x)\cos x + v_1(x)\sin x + v_2(x)\cos x = \cot x$$

$$v_1'(x)\cos x - v_2'(x)\sin x = \cot x$$

So we have the system

$$(\sin x)\, v_1'(x) + (\cos x)\, v_2'(x) = 0$$

$$(\cos x)\, v_1'(x) + (-\sin x)\, v_2'(x) = \cot x$$

solving the system:

$$v_1'(x) = \frac{\begin{vmatrix} 0 & \cos x \\ \cot x & -\sin x \end{vmatrix}}{\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}} = \frac{-\cos x \cot x}{-\sin^2 x - \cos^2 x} = \frac{-\cos x \left(\frac{\cos x}{\sin x}\right)}{-1}$$

$$= \frac{\cos^2 x}{\sin x}$$

(p.169 #1 continued)

$$v_2'(x) = \frac{\begin{vmatrix} \sin x & 0 \\ \cos x & \cot x \end{vmatrix}}{\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}} = \frac{\sin x \cot x}{-1}$$

$$= -\sin x\left(\frac{\cos x}{\sin x}\right)$$

$$= -\cos x$$

So we have found

$$v_1'(x) = \frac{\cos^2 x}{\sin x}, \quad v_2'(x) = -\cos x$$

Then

$$v_1(x) = \int \frac{\cos^2 x}{\sin x}\,dx = \int \frac{1-\sin^2 x}{\sin x}\,dx$$

$$= \int (\csc x - \sin x)\,dx$$

$$= \ln|\csc x - \cot x| + \cos x + c_3$$

$$v_2(x) = \int -\cos x\,dx = -\sin x + c_4$$

Hence $y_p = (\ln|\csc x - \cot x| + \cos x + c_3)\sin x + (-\sin x + c_4)\cos x$

$$= \sin x \ln|\csc x - \cot x| + \sin x \cos x + c_3 \sin x - \sin x \cos x + c_4 \cos x$$

$$= (\sin x)\ln|\csc x - \cot x| + c_3 \sin x + c_4 \cos x$$

Since we require only one particular soln., we may choose any value for c_3 & c_4, say both 0, and

$$y_p = (\sin x)\ln|\csc x - \cot x|$$

Then $y = y_c + y_p = c_1 \sin x + c_2 \cos x + (\sin x)\ln|\csc x - \cot x|$

$$y'' + y = \sec x$$

The complementary soln. is obtained the same as #1 :

$$y_c = c_1 \sin x + c_2 \cos x$$

p.169
#3

from #1

$$\begin{cases} y_p = v_1(x) \sin x + v_2(x) \cos x \\ y_p' = v_1(x) \cos x - v_2(x) \sin x \\ y_p'' = v_1'(x) \cos x - v_1(x) \sin x - v_2'(x) \sin x - v_2(x) \cos x \end{cases}$$

The diff. eqn. becomes

$$v_1'(x) \cos x - v_2'(x) \sin x = \sec x$$

so we have the system

$$(\sin x)\, v_1'(x) + (\cos x)\, v_2'(x) = 0$$
$$(\cos x)\, v_1'(x) + (-\sin x)\, v_2'(x) = \sec x$$

solving the system :

$$v_1'(x) = \frac{\begin{vmatrix} 0 & \cos x \\ \sec x & -\sin x \end{vmatrix}}{\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}} = \frac{-\cos x \sec x}{-\sin^2 x - \cos^2 x} = \frac{-1}{-1} = 1$$

$$v_2'(x) = \frac{\begin{vmatrix} \sin x & 0 \\ \cos x & \sec x \end{vmatrix}}{\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}} = \frac{\sin x \sec x}{-1} = -\tan x$$

integrating : $v_1(x) = \int 1\, dx = x + c_3$

$$v_2(x) = \int -\tan x\, dx = \ln|\cos x| + c_4$$

letting $c_3 = c_4 = 0$, $v_1(x) = x$, $v_2(x) = \ln|\cos x|$

$$y_p = x \sin x + (\cos x) \ln|\cos x|$$

$$y = y_c + y_p = c_1 \sin x + c_2 \cos x + x \sin x + (\cos x) \ln|\cos x|$$

$$y'' + 4y' + 5y = e^{-2x}\sec x$$

The characteristic eqn. is

$$m^2 + 4m + 5 = 0$$

p. 170 #7

$$m = \frac{-4 \pm \sqrt{4^2 - 4(1)(5)}}{2(1)} = \frac{-4 \pm \sqrt{-4}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i$$

$$y_c = C_1 e^{-2x}\sin x + C_2 e^{-2x}\cos x$$

we assume a particular soln. of the form

$$y_p = v_1(x) e^{-2x}\sin x + v_2(x) e^{-2x}\cos x$$

Then $y_p' = v_1(x)e^{-2x}\cos x - 2v_1(x)e^{-2x}\sin x - v_2(x)e^{-2x}\sin x - 2v_2(x)e^{-2x}\cos x$
$+ \left(v_1'(x)e^{-2x}\sin x + v_2'(x)e^{-2x}\cos x\right)$

Letting $v_1'(x)e^{-2x}\sin x + v_2'(x)e^{-2x}\cos x = 0$, we have

$$y_p' = v_1(x)e^{-2x}\cos x - 2v_1(x)e^{-2x}\sin x - v_2(x)e^{-2x}\sin x - 2v_2(x)e^{-2x}\cos x$$

$$\begin{aligned} y_p'' = & -v_1(x)e^{-2x}\sin x - 2v_1(x)e^{-2x}\cos x + v_1'(x)e^{-2x}\cos x \\ & - 2v_1(x)e^{-2x}\cos x + 4v_1(x)e^{-2x}\sin x - 2v_1'(x)e^{-2x}\sin x \\ & - v_2(x)e^{-2x}\cos x + 2v_2(x)e^{-2x}\sin x - v_2'(x)e^{-2x}\sin x \\ & + 2v_2(x)e^{-2x}\sin x + 4v_2(x)e^{-2x}\cos x - 2v_2'(x)e^{-2x}\cos x \end{aligned}$$

substituting y_p'', y_p', and y_p into the original diff. eqn.

$$\begin{aligned} & \left[\begin{aligned} & -v_1(x)e^{-2x}\sin x - 2v_1(x)e^{-2x}\cos x + v_1'(x)e^{-2x}\cos x \\ & -2v_1(x)e^{-2x}\cos x + 4v_1(x)e^{-2x}\sin x - 2v_1'(x)e^{-2x}\sin x \\ & - v_2(x)e^{-2x}\cos x + 2v_2(x)e^{-2x}\sin x - v_2'(x)e^{-2x}\sin x \\ & + 2v_2(x)e^{-2x}\sin x + 4v_2(x)e^{-2x}\cos x - 2v_2'(x)e^{-2x}\cos x \end{aligned} \right] \\ & + \left(4v_1(x)e^{-2x}\cos x - 8v_1(x)e^{-2x}\sin x - 4v_2(x)e^{-2x}\sin x - 8v_2(x)e^{-2x}\cos x\right) \\ & + \left(5v_1(x)e^{-2x}\sin x + 5v_2(x)e^{-2x}\cos x\right) \\ & = e^{-2x}\sec x \end{aligned}$$

$\left(\begin{matrix} p.170 \\ \#7 \end{matrix} \quad \text{cont.} \right)$

which simplifies to

$$v_1'(x) e^{-2x} \cos x - 2 v_1'(x) e^{-2x} \sin x - v_2'(x) e^{-2x} \sin x$$
$$- 2 v_2'(x) e^{-2x} \cos x = e^{-2x} \sec x$$

so we have the system

$$(e^{-2x} \sin x)\, v_1'(x) + (e^{-2x} \cos x)\, v_2'(x) = 0$$
$$(e^{-2x} \cos x - 2 e^{-2x} \sin x)\, v_1'(x) + (-e^{-2x} \sin x - 2 e^{-2x} \cos x)\, v_2'(x) = e^{-2x} \sec x$$

solving the system:

$$v_1'(x) = \frac{\begin{vmatrix} 0 & e^{-2x} \cos x \\ e^{-2x} \sec x & -e^{-2x} \sin x - 2 e^{-2x} \cos x \end{vmatrix}}{\begin{vmatrix} e^{-2x} \sin x & e^{-2x} \cos x \\ e^{-2x} \cos x - 2 e^{-2x} \sin x & -e^{-2x} \sin x - 2 e^{-2x} \cos x \end{vmatrix}}$$

$$= \frac{-e^{-4x} \cos x \sec x}{-e^{-4x} \sin^2 x - 2 e^{-4x} \sin x \cos x - e^{-4x} \cos^2 x + 2 e^{-4x} \sin x \cos x} = \frac{-e^{-4x}}{-e^{-4x}} = 1$$

$$v_2'(x) = \frac{\begin{vmatrix} e^{-2x} \sin x & 0 \\ e^{-2x} \cos x - 2 e^{-2x} \sin x & e^{-2x} \sec x \end{vmatrix}}{-e^{-4x}} = \frac{e^{-4x} \sin x \sec x}{-e^{-4x}} = -\tan x$$

integrating : $v_1(x) = x + c_3 \qquad v_2(x) = \ln|\cos x| + c_4$

Letting $c_3 = c_4 = 0$

$$y_p = x e^{-2x} \sin x + (e^{-2x} \cos x) \ln|\cos x|$$

$$y = y_c + y_p$$
$$= c_1 e^{-2x} \sin x + c_2 e^{-2x} \cos x + x e^{-2x} \sin x + (e^{-2x} \cos x) \ln|\cos x|$$

$$y'' + 3y' + 2y = \frac{1}{1+e^x}$$

The characteristic eqn. is

p. 170
#13

$$m^2 + 3m + 2 = 0$$

$$(m+2)(m+1) = 0$$

$$m = -2 \qquad m = -1$$

$$y_c = c_1 e^{-2x} + c_2 e^{-x}$$

we look for a particular soln. of the form

$$y_p = v_1(x) e^{-2x} + v_2(x) e^{-x}$$

$$y_p' = -2v_1(x) e^{-2x} - v_2(x) e^{-x} + \left(v_1'(x) e^{-2x} + v_2'(x) e^{-x}\right)$$

Letting $v_1'(x) e^{-2x} + v_2'(x) e^{-x} = 0$, we have

$$y_p' = -2 v_1(x) e^{-2x} - v_2(x) e^{-x}$$

$$y_p'' = -2v_1'(x) e^{-2x} + 4v_1(x) e^{-2x} - v_2'(x) e^{-x} + v_2(x) e^{-x}$$

substituting y_p'', y_p', and y_p into the original diff. eqn.

$$-2v_1'(x) e^{-2x} + 4v_1(x) e^{-2x} - v_2'(x) e^{-x} + v_2(x) e^{-x}$$
$$-6v_1(x) e^{-2x} - 3v_2(x) e^{-x}$$
$$+2v_1(x) e^{-2x} + 2v_2(x) e^{-x} = \frac{1}{1+e^x}$$

$$-2v_1'(x) e^{-2x} - v_2'(x) e^{-x} = \frac{1}{1+e^x}$$

so we have the system of eqns.

$$\left(e^{-2x}\right) v_1'(x) + \left(e^{-x}\right) v_2'(x) = 0$$

$$\left(-2e^{-2x}\right) v_1'(x) + \left(-e^{-x}\right) v_2'(x) = \frac{1}{1+e^x}$$

Solving the system:

$$v_1'(x) = \frac{\begin{vmatrix} 0 & e^{-x} \\ \frac{1}{1+e^x} & -e^{-x} \end{vmatrix}}{\begin{vmatrix} e^{-2x} & e^{-x} \\ -2e^{-2x} & -e^{-x} \end{vmatrix}} = \frac{-\frac{1}{e^x}\left(\frac{1}{1+e^x}\right)}{-e^{-3x} + 2e^{-3x}}$$

$$= -e^{2x}\left(\frac{1}{1+e^x}\right) = \frac{-e^{2x}}{e^x+1}$$

(p.170 #13 cont.)

$$v_2'(x) = \frac{\begin{vmatrix} e^{-2x} & 0 \\ -2e^{-2x} & \frac{1}{1+e^x} \end{vmatrix}}{e^{-3x}} = \frac{e^{-2x}\left(\frac{1}{1+e^x}\right)}{e^{-3x}}$$

$$= \frac{e^x}{e^x+1}$$

integrating:

$$v_1(x) = \int \frac{-e^{2x}}{e^x+1}\,dx \qquad \text{Let } u = e^x+1$$
$$du = e^x\,dx$$

$$= -\int \frac{(u-1)}{u}\,du$$

$$= -\int \left(1 - \frac{1}{u}\right)du = -[u - \ln u] + c_3'$$

$$= \ln(e^x+1) - (e^x+1) + c_3'$$

$$= \ln(e^x+1) - e^x + c_3$$

$$v_2(x) = \int \frac{e^x}{e^x+1}\,dx = \ln(e^x+1) + c_4$$

Letting $c_3 = c_4 = 0$

$$y_p = (e^{-2x})\left[\ln(e^x+1) - e^x\right] + (e^{-x})\left[\ln(e^x+1)\right]$$
$$= (e^{-2x})\ln(e^x+1) - e^{-x} + (e^{-x})\ln(e^x+1)$$

$$y = y_c + y_p$$
$$= c_1 e^{-2x} + c_2 e^{-x} + (e^{-2x})\ln(e^x+1) - e^{-x} + (e^{-x})\ln(e^x+1)$$

$$y'' + 3y' + 2y = \frac{e^{-x}}{x}$$

p.170 #17

The characteristic eqn. is

$$m^2 + 3m + 2 = 0$$

$$(m+1)(m+2) = 0$$

$$m = -1 \qquad m = -2$$

$$y_c = c_1 e^{-x} + c_2 e^{-2x}$$

we look for a particular soln. of the form

$$y_p = v_1(x) e^{-x} + v_2(x) e^{-2x}$$

$$y_p' = -v_1(x) e^{-x} - 2 v_2(x) e^{-2x} + (v_1'(x) e^{-x} + v_2'(x) e^{-2x})$$

Letting $(v_1'(x) e^{-x} + v_2'(x) e^{-2x}) = 0$

$$y_p' = -v_1(x) e^{-x} - 2 v_2(x) e^{-2x}$$

$$y_p'' = -v_1'(x) e^{-x} + v_1(x) e^{-x} - 2 v_2'(x) e^{-2x} + 4 v_2(x) e^{-2x}$$

substituting y_p'', y_p', y_p into the original diff. eqn.

$$-v_1'(x) e^{-x} + v_1(x) e^{-x} - 2 v_2'(x) e^{-2x} + 4 v_2(x) e^{-2x}$$

$$-3 v_1(x) e^{-x} - 6 v_2(x) e^{-2x}$$

$$+ 2 v_1(x) e^{-x} + 2 v_2(x) e^{-2x} = \frac{e^{-x}}{x}$$

$$-v_1'(x) e^{-x} - 2 v_2'(x) e^{-2x} = \frac{e^{-x}}{x}$$

so we have the system of eqns:

$$(e^{-x}) v_1'(x) + (e^{-2x}) v_2'(x) = 0$$

$$(-e^{-x}) v_1'(x) + (-2e^{-2x}) v_2'(x) = \frac{e^{-x}}{x}$$

solving the system:

$$v_1'(x) = \frac{\begin{vmatrix} 0 & e^{-2x} \\ \frac{e^{-x}}{x} & -2e^{-2x} \end{vmatrix}}{\begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix}} = \frac{-\frac{e^{-3x}}{x}}{-2e^{-3x} + e^{-3x}}$$

$$= 1/x$$

(p. 170 #17 cont.)

$$v_2'(x) = \frac{\begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & \frac{e^{-x}}{x} \end{vmatrix}}{-e^{-3x}} = \frac{\frac{e^{-2x}}{x}}{-e^{-3x}} = \frac{-e^{x}}{x}$$

integrating:

$$v_1(x) = \int \frac{1}{x}\,dx = \ln|x| + c_3$$

$$v_2(x) = \int \frac{-e^{x}}{x}\,dx + c_4$$

(try parts, substitution, etc. + cannot integrate)

letting $c_3 = c_4 = 0$

$$y_p = (e^{-x})\ln|x| + (e^{-2x})\int \frac{-e^{x}}{x}\,dx$$

$$y = y_c + y_p$$

$$= c_1 e^{-x} + c_2 e^{-2x} + (e^{-x})\ln|x| - (e^{-2x})\int \frac{e^{x}}{x}\,dx$$

$wt = 12 \text{ lb}$

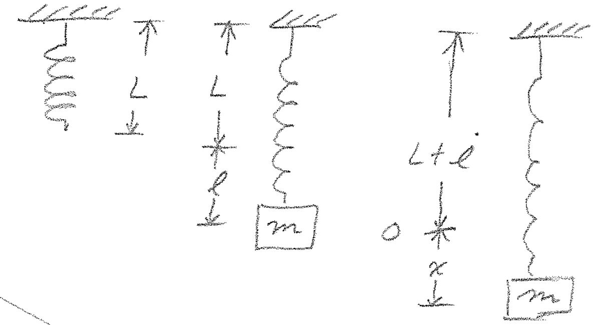
$12 \text{ lb} = k\left(\frac{1.5}{12} \text{ ft}\right)$

p.197

#1

$k = 96 \text{ lb/ft}$

at $t = 0$ $x = \frac{2}{12}$, $v = 0$



Free, undamped motion

$$m x'' + k x = 0$$

$$\frac{12}{32} x'' + 96 x = 0$$

$$x'' + 256 x = 0$$

characteristic eqn.

$$m^2 + 256 = 0$$

$$m = \pm\sqrt{-256} = \pm 16i$$

General soln. $x = e^{0t}(c_1 \sin 16t + c_2 \cos 16t)$

$$x = c_1 \sin 16t + c_2 \cos 16t$$

applying the initial conditions $x(0) = \frac{1}{6}$, $x'(0) = 0$

$$\frac{1}{6} = c_2$$

$$x' = 16c_1 \cos 16t - 16c_2 \sin 16t$$

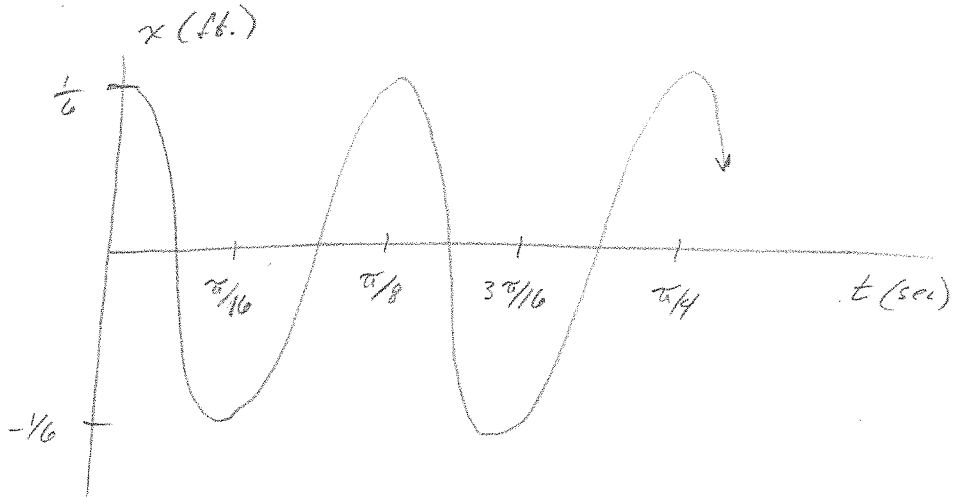
$$0 = 16c_1 \rightarrow c_1 = 0$$

$x = \frac{1}{6} \cos 16t$

Amplitude $\frac{1}{6}$ ft

period $\frac{2\pi}{16} = \frac{\pi}{8}$ sec.

frequency $\frac{8}{\pi}$ $\frac{\text{oscillations}}{\text{sec.}}$

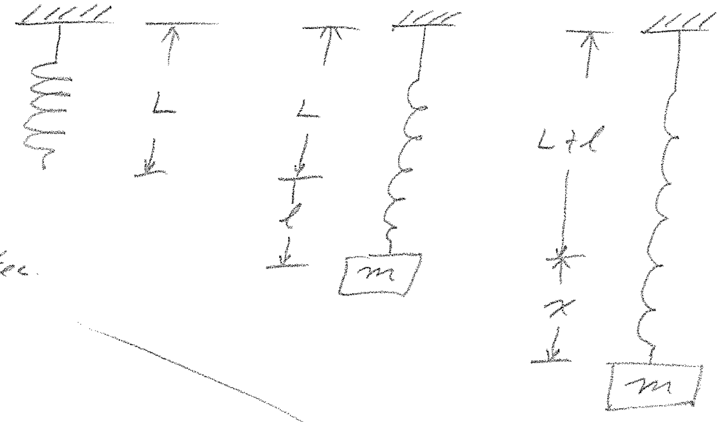


$$wt = 4\text{ lb}$$

$$4\text{ lb} = k\left(\tfrac{6}{12}\text{ ft}\right)$$

p. 197 #5

$$k = 8\text{ lb/ft.}$$



at $t=0$, $x=0$, $v = 2$ ft/sec

$$m x'' + k x = 0$$

$$\frac{4}{32} x'' + 8x = 0$$

$$x'' + 64x = 0$$

The characteristic eqn. is

$$m^2 + 64 = 0$$

$$m = \pm 8i$$

$$x = c_1 \sin 8t + c_2 \cos 8t$$

applying the initial conditions $x(0) = 0$, $x'(0) = 2$

$$0 = c_2$$

$$x' = 8c_1 \cos 8t - 8c_2 \sin 8t$$

$$2 = 8c_1 \longrightarrow c_1 = 1/4$$

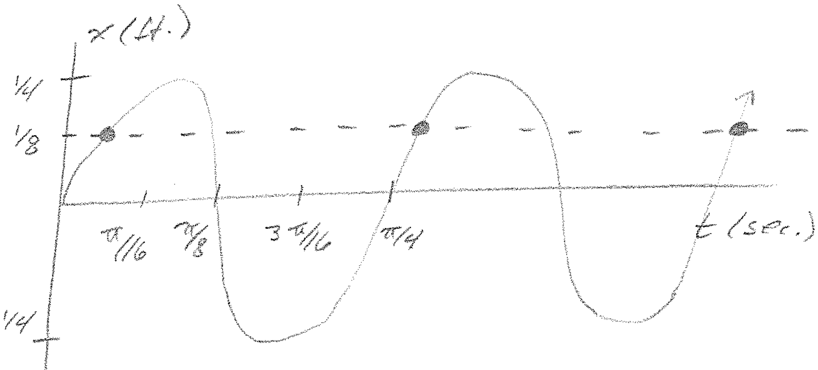
a.) $x = \frac{1}{4} \sin 8t$

$$v = x' = 2\cos 8t$$

b) Amplitude $= 1/4$ ft

period $= \dfrac{2\pi}{8} = \pi/4$ sec.

frequency $= 4/\pi \ \dfrac{\text{oscillations}}{\text{sec.}}$



c) 1.5 in $= 1/8$ ft.

$$1/8 = 1/4 \sin 8t$$

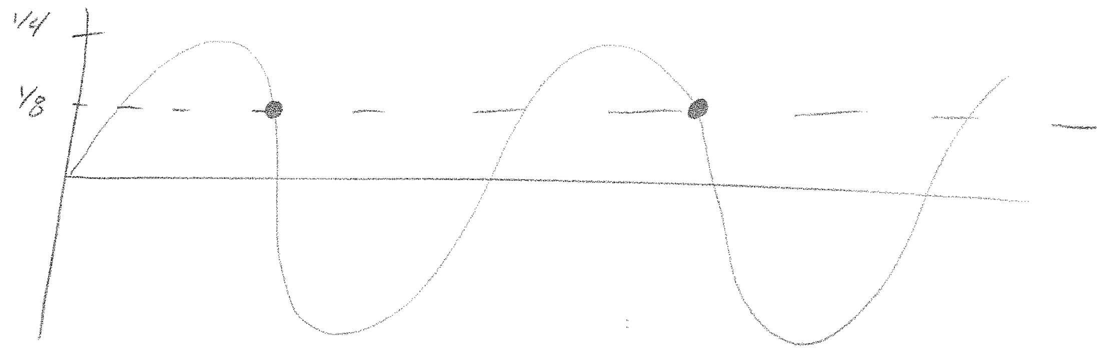
$$\sin 8t = 1/2$$

$$8t = \pi/6,\ 2\pi + \pi/6,\ 4\pi + \pi/6, \ldots$$

$$t = \pi/48,\ \frac{n\pi}{4} + \pi/48 \text{ sec.}, \quad n = 1, 2, \ldots$$

(p. 197 #5 continued)

(d)



$\sin 8t = 1/2$

$8t = 5\pi/6 \;,\; 2\pi + 5\pi/6 \;,\; 4\pi + 5\pi/6 \;, \ldots$

$t = 5\pi/48 \;,\; \frac{n\pi}{4} + 5\pi/48 \;,\quad n = 1, 2, \ldots$

wt 8 lb

$F = ks$

$8\,lb = k\,(0.4\,ft)$

p. 208
#1

$k = 20\ lb/ft$

at $t = 0$, $x = \frac{6}{12} = \frac{1}{2}\ ft$, $v = 0$

damping force $= 2x'$

Free, damped motion

$$m x'' + a x' + k x = 0$$

$$\frac{8}{32} x'' + 2x' + 20x = 0$$

(a) $$x'' + 8x' + 80x = 0, \quad x(0) = \tfrac{1}{2}, \ x'(0) = 0$$

Characteristic Eqn.

$$m^2 + 8m + 80 = 0$$

$$m = \frac{-8 \pm \sqrt{(8)^2 - 4(1)(80)}}{2(1)} = \frac{-8 \pm 16i}{2} = -4 \pm 8i$$

General soln. $x = e^{-4t}(c_1 \sin 8t + c_2 \cos 8t)$ $\boxed{\text{Underdamped}}$

applying initial conditions

$$\frac{1}{2} = c_2$$

$$x' = e^{-4t}(8c_1 \cos 8t - 8c_2 \sin 8t) - 4e^{-4t}(c_1 \sin 8t + c_2 \cos 8t)$$

$$0 = 8c_1 - 4c_2$$

$$0 = 8c_1 - 4(\tfrac{1}{2}) \rightarrow c_1 = \tfrac{1}{4}$$

b) $x = e^{-4t}(\tfrac{1}{4} \sin 8t + \tfrac{1}{2} \cos 8t)$

$x = \tfrac{1}{2} e^{-4t}(\tfrac{1}{2} \sin 8t + \cos 8t)$

Trig. Reduction identity

$x = \tfrac{1}{2} e^{-4t} \sqrt{(\tfrac{1}{2})^2 + (1)^2} \cos(8t + \phi)$

c) $x = \frac{\sqrt{5}}{4} e^{-4t} \cos(8t + \phi)$

(p. 218 #1 cont.)

where $\dfrac{1/2}{\sqrt{(1/2)^2+(1)^2}} = -\sin\phi$ and $\dfrac{1}{\sqrt{(1/2)^2+(1)^2}} = \cos\phi$

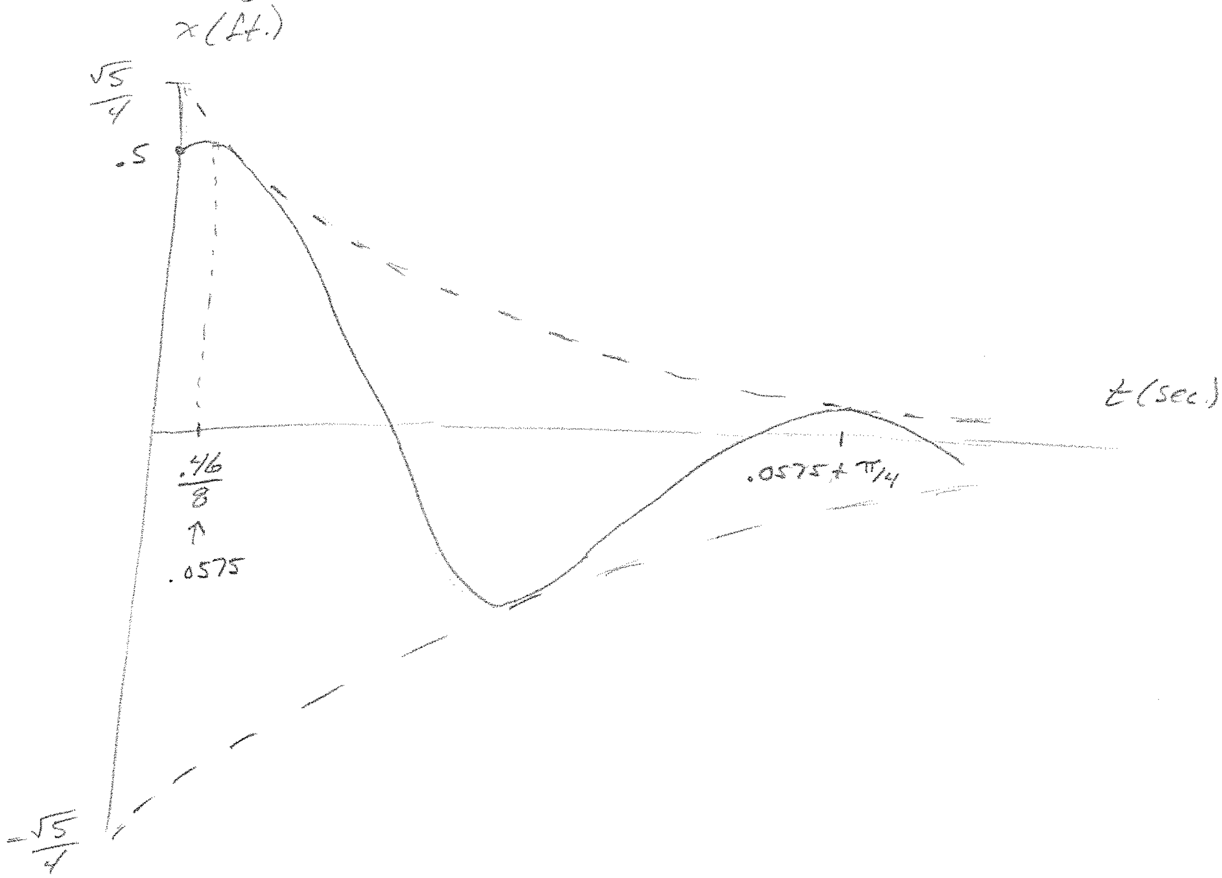
$$\underbrace{\sin\phi = \frac{-1/2}{\frac{\sqrt{5}}{2}} = \frac{-1}{\sqrt{5}} \qquad \cos\phi = \frac{1}{\frac{\sqrt{5}}{2}} = \frac{2}{\sqrt{5}}}_{\phi \text{ in IV th Quadr.}}$$

$$\phi \approx -.46$$

$$x = \frac{\sqrt{5}}{4} e^{-4t} \cos(8t - .46)$$

(d) quasi - period $\dfrac{2\pi}{8} = \pi/4$ sec.

(e)



8 lb wt.

$F = ks$

p. 208 #3

$8 \text{ lb} = k\left(\frac{6}{12} \text{ ft}\right)$

$k = 16 \text{ lb/ft}$

at $t=0$, $x = \frac{9}{12} = 3/4$ ft., $v = 0$

damping force $4x'$

Free, damped motion

$mx'' + ax' + kx = 0$

$\frac{8}{32}x'' + 4x' + 16x = 0$

$x'' + 16x' + 64x = 0$

characteristic eqn.

$m^2 + 16m + 64 = 0$

$(m+8)(m+8) = 0$

$m = -8 \quad m = -8$

Critically damped

general soln. $x = c_1 e^{-8t} + c_2 t e^{-8t}$

applying the initial conditions $x(0) = 3/4$, $x'(0) = 0$

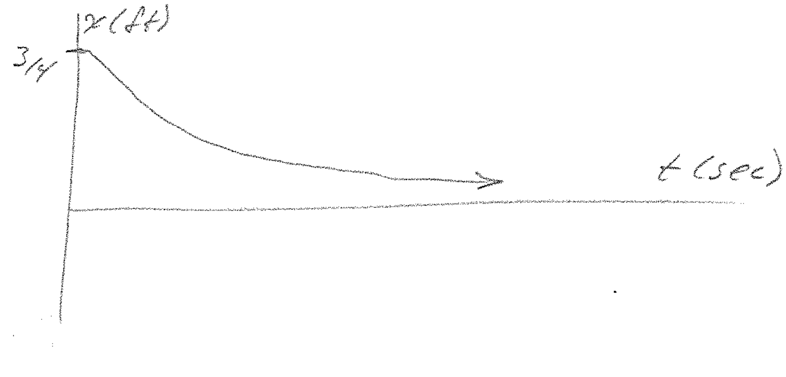
$3/4 = c_1$

$x' = -8c_1 e^{-8t} + c_2 e^{-8t} - 8c_2 t e^{-8t}$

$0 = -8c_1 + c_2$

$0 = -8(3/4) + c_2 \rightarrow c_2 = 6$

$x = 3/4 e^{-8t} + 6te^{-8t}$



since

$x' = -6e^{-8t} + 6e^{-8t} - 48te^{-8t}$

$= -48te^{-8t}$ always negative for $t>0$, monotonic decreasing wt. 16 lb $\qquad$ damping force $10x'$

$F = kS$

$16 = k\left(\frac{6}{12}\right)$

p.209
#4

$k = 32$ lb/ft

at $t=0$, $x = \frac{3}{12} = \frac{1}{4}$ ft., $v=0$

Free, damped motion

$$m x'' + a x' + k x = 0$$

$$\frac{16}{32} x'' + 10x' + 32x = 0$$

$$x'' + 20x' + 64x = 0$$

characteristic eqn.

$$m^2 + 20m + 64 = 0$$

$$(m+4)(m+16) = 0$$

$$m = -4 \qquad m = -16$$

$\boxed{\text{Overdamped}}$

General soln. $x = C_1 e^{-4t} + C_2 e^{-16t}$

applying the initial conditions $x(0) = 1/4$, $x'(0) = 0$

$$1/4 = C_1 + C_2$$

$$x' = -4C_1 e^{-4t} - 16C_2 e^{-16t}$$

$$0 = -4C_1 - 16C_2 \longrightarrow C_1 = -4C_2$$

$$1/4 = -4C_2 + C_2$$

$$C_2 = -1/12$$

$$C_1 = 1/3$$

$$x = \tfrac{1}{3}e^{-4t} - \tfrac{1}{12}e^{-16t}$$

does it pass thru axis?

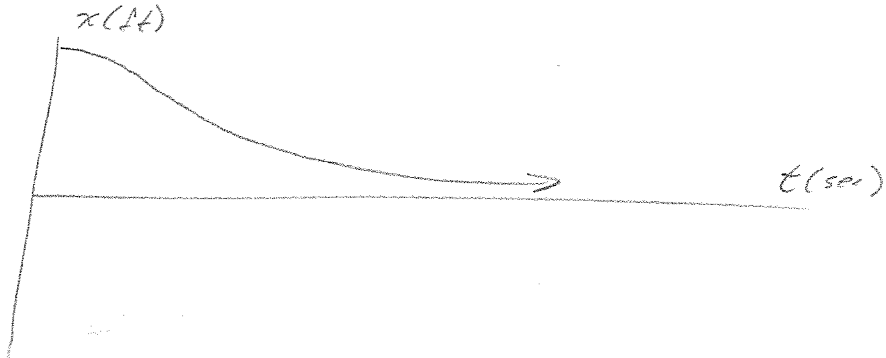
$0 = \frac{1}{3}e^{-4t} - \frac{1}{12}e^{-16t}$

$4e^{-4t} = e^{-16t}$

$4 = e^{-12t}$

$-12t = \ln 4$

$t = -.116$ $\qquad$ No



$F = ks$ $\qquad$ $wt = 4 \text{ lb}$

$20 = k\left(\frac{6}{12}\right)$ $\qquad$ damping force $= 2x'$

p. 209 $\qquad$ $k = 40 \text{ lb/ft}$

#7

at $t=0$, $x = \frac{8}{12}$ ft, $v=0$
$= 2/3$

Free, damped motion

$$mx'' + ax' + kx = 0$$

$$\frac{4}{32}x'' + 2x' + 40x = 0$$

$$x'' + 16x' + 320x = 0$$

characteristic eqn.

$$m^2 + 16m + 320 = 0$$

$$m = \frac{-16 \pm \sqrt{(16)^2 - 4(1)(320)}}{2(1)} = \frac{-16 \pm 32i}{2} = -8 \pm 16i$$

General soln. $x = e^{-8t}(c_1 \sin 16t + c_2 \cos 16t)$

applying initial conditions $x(0) = 2/3$, $x'(0) = 0$

$$2/3 = c_2$$

$$x' = e^{-8t}(16c_1 \cos 16t - 16c_2 \sin 16t) - 8e^{-8t}(c_1 \sin 16t + c_2 \cos 16t)$$

$$0 = 16c_1 - 8c_2$$

$$0 = 16c_1 - 8(2/3) \longrightarrow c_1 = 1/3$$

$$x = e^{-8t}\left(1/3 \sin 16t + 2/3 \cos 16t\right)$$

$$x = 1/3\, e^{-8t}(\sin 16t + 2\cos 16t)$$

$$x = 1/3\, e^{-8t}\sqrt{(1)^2 + (2)^2}\cos(16t + \phi)$$

$$x = \frac{\sqrt{5}}{3}e^{-8t}\cos(16t + \phi)$$

(p. 209 #7 cont.)

where

$$\underbrace{\frac{1}{\sqrt{5}} = -\sin\phi \qquad \text{and} \qquad \frac{2}{\sqrt{5}} = \cos\phi}$$

ϕ in quadr. IV

$\phi \approx -.46$

$$x = \frac{\sqrt{5}}{3} e^{-8t} \cos(16t - .46)$$

(c) wt. passes thru equilib. position when $x = 0$

i.e. $0 = \frac{\sqrt{5}}{3} e^{-8t} \cos(16t - .46)$

$\cos(16t - .46) = 0$

this occurs first at

$16t - .46 = \pi/2$

$t \approx .127$

wt. 4 lb

damping force ax'

$F = kS$

$4 = k\left(\frac{8}{12}\right)$

p. 210
#9

$k = 6$ lb/ft

at $t=0$, $x = x_0$, $v = 0$

Free, damped motion

$$m x'' + a x' + k x = 0$$

$$\frac{4}{32} x'' + a x' + 6x = 0$$

$$x'' + 8a x' + 48 x = 0$$

characteristic eqn.

$$m^2 + 8am + 48 = 0 \leftarrow$$

<u>critically</u> <u>damped</u> if this is a perfect square
i.e.
$8a = 2\sqrt{48}$
$a = \frac{8\sqrt{3}}{8} = \sqrt{3}$

$$m = \frac{-8a \pm \sqrt{(8a)^2 - 4(1)(48)}}{2(1)}$$

<u>Under damped</u> if m has an imaginary component.

i.e. $(8a)^2 - 4(1)(48) < 0$

$64a^2 - 192 < 0$

$a^2 < 3$

$a < \sqrt{3}$ (recall a is positive)

<u>over damped</u> otherwise.

i.e. $a > \sqrt{3}$

wt. 6 lb.
spring constant
$k = 27$ lb/ft

Damping negligible
$\rightarrow a = 0$

p. 217 #1

$F(t) = 12 \cos 20t$

at $t = 0,\ x = 0,\ v = 0$

$$m x'' + \cancel{a x'}_{0} + kx = F(t)$$

$$\frac{6}{32} x'' + 27x = 12 \cos 20t$$

$$(*) \quad x'' + 144x = 64 \cos 20t$$

The corresponding homog. eqn.

$x'' + 144x = 0$ has characteristic eqn.

$$m^2 + 144 = 0$$

$$m = \pm 12i$$

so the complementary soln. is

$$x_c = e^{0t}(c_1 \sin 12t + c_2 \cos 12t)$$

$$x_c = c_1 \sin 12t + c_2 \cos 12t$$

Using the method of undetermined coefficients:

The nonhomogeneous term $64 \cos 20t$ has UC set

$$S_1 = \{\sin 20t,\ \cos 20t\}$$

Since none of these elements belong to the solution x_c (look again... $\sin \mathbf{12t}$, $\sin \mathbf{20t}$), we do not modify S_1

$$x_p = A \sin 20t + B \cos 20t$$

$$x_p' = 20A \cos 20t - 20B \sin 20t$$

$$x_p'' = -400A \sin 20t - 400B \cos 20t$$

$(*)$ becomes $-400A \sin 20t - 400B \cos 20t + 144(A \sin 20t + B \cos 20t) = 64 \cos 20t$

$$-256A \sin 20t - 256B \cos 20t = 64 \cos 20t$$

$$-256A = 0 \longrightarrow A = 0$$

$$-256B = 64 \longrightarrow B = -1/4$$

(p.217 #1 cont.)

$$x_p = -\tfrac{1}{4}\cos 20t$$

$$x = x_c + x_p = c_1 \sin 12t + c_2 \cos 12t - \tfrac{1}{4}\cos 20t$$

Applying the initial conditions $x(0)=0$, $x'(0)=0$

$$0 = c_2 - \tfrac{1}{4} \rightarrow c_2 = \tfrac{1}{4}$$

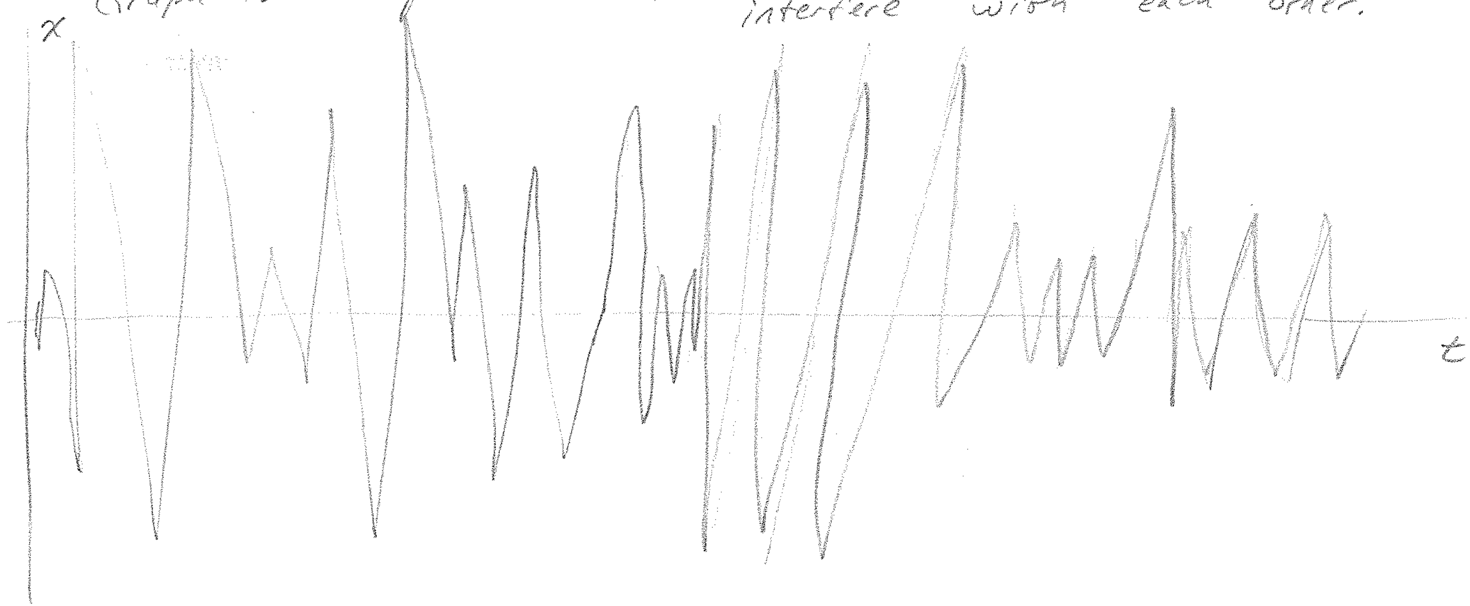
$$x' = 12c_1 \cos 12t - 12c_2 \sin 12t + 5 \sin 20t$$

$$0 = 12c_1 \rightarrow c_1 = 0$$

$$x = \tfrac{1}{4}\cos 12t - \tfrac{1}{4}\cos 20t$$

(No transient terms here)

Graph is very erratic, since the two cosine functions interfere with each other.



wt 16 lb

damping force $4x'$

$F = ks$

$16 = k(0.4 ft)$

$k = 40$ lb/ft

at $t=0$, $x=0$, $v=0$

p.217 #2

$F(t) = 40 \cos 16t$

$m x'' + a x' + k x = F(t)$

$\frac{16}{32} x'' + 4x' + 40x = 40 \cos 16t$

$\circledast \quad x'' + 8x' + 80x = 80 \cos 16t$

The corresponding homog. eqn. $x'' + 8x' + 80x = 0$ has charact. eqn.

$m^2 + 8m + 80 = 0$

$m = \frac{-8 \pm \sqrt{(8)^2 - 4(1)(80)}}{2(1)} = \frac{-8 \pm \sqrt{-256}}{2} = \frac{-8 \pm 16i}{2} = -4 \pm 8i$

the complementary soln. is

$x_c = e^{-4t}(c_1 \sin 8t + c_2 \cos 8t)$

$= c_1 e^{-4t} \sin 8t + c_2 e^{-4t} \cos 8t$

Using <u>Undetermined coeff's.</u> The non homog. term

$80 \cos 16t$ has UC set

$S_1 = \{\sin 16t, \cos 16t\}$

none are solns. of the homog. eqn.

$x_p = A \sin 16t + B \cos 16t$

$x_p' = 16A \cos 16t - 16B \sin 16t$

$x_p'' = -256 A \sin 16t - 256 B \cos 16t$

(p. 217 #2 cont.)

substituting into ⊗

$$(-256A\sin 16t - 256B\cos 16t) + 8(16A\cos 16t - 16B\sin 16t) + 80(A\sin 16t + B\cos 16t)$$
$$= 80\cos 16t$$

$$(-176A - 128B)\sin 16t + (-176B + 128A)\cos 16t = 80\cos 16t$$

(i) $-176A - 128B = 0 \rightarrow A = -\frac{128}{176}B = \frac{-8}{11}B$

(ii) $-176B + 128A = 80$

(ii) $-176B + 128\left(\frac{-8}{11}B\right) = 80$

$$-1936B - 1024B = 880$$
$$-2960B = 880$$
$$B = -\frac{880}{2960} = \frac{-11}{37}$$

$$A = \frac{-8}{11}\left(\frac{-11}{37}\right) = \frac{8}{37}$$

$$x_p = \frac{8}{37}\sin 16t - \frac{11}{37}\cos 16t$$

$$x = x_c + x_p = C_1 e^{-4t}\sin 8t + C_2 e^{-4t}\cos 8t + \frac{8}{37}\sin 16t - \frac{11}{37}\cos 16t$$

Applying the initial conditions $x(0)=0$, $x'(0)=0$

$$0 = C_2 - \frac{11}{37} \rightarrow C_2 = \frac{11}{37}$$

$$x' = 8C_1 e^{-4t}\cos 8t - 4C_1 e^{-4t}\sin 8t - 8C_2 e^{-4t}\sin 8t - 4C_2 e^{-4t}\cos 8t + \frac{128}{37}\cos 16t + \frac{176}{37}\sin 16t$$

$x'(0) = 0$:

$$0 = 8C_1 - 4\left(\frac{11}{37}\right) + \frac{128}{37} \rightarrow C_1 = \frac{-21}{74}$$

$$x = -\frac{21}{74}e^{-4t}\sin 8t + \frac{11}{37}e^{-4t}\cos 8t + \frac{8}{37}\sin 16t - \frac{11}{37}\cos 16t$$

wt. 10 lb
spr. constant
$k = 20$ lb/ft.
$F(t) = 10 \cos 8t$

damping force $5x'$

at $t=0$, $x=0$, $v=0$

p. 217
#3

$$mx'' + ax' + kx = F(t)$$

$$\frac{10}{32} x'' + 5x' + 20x = 10 \cos 8t$$

$$x'' + 16x' + 64x = 32 \cos 8t$$

The corresponding homog. eqn.

$x'' + 16x' + 64x = 0$ has charact. eqn.

$$m^2 + 16m + 64 = 0$$

$$(m+8)(m+8) = 0$$

$$m = -8 \qquad m = -8$$

The complementary soln. is

$$x_c = c_1 e^{-8t} + c_2 t e^{-8t}$$

Using the method of undetermined coefficients:

The nonhomog. term $32 \cos 8t$ has UC set

$$S_1 = \{\sin 8t, \cos 8t\}$$

since neither elt. of S_1 is a soln. of the homog. eqn., we do not modify S_1.

$$x_p = A \sin 8t + B \cos 8t$$

$$x_p' = 8A \cos 8t - 8B \sin 8t$$

$$x_p'' = -64A \sin 8t - 64B \cos 8t$$

(p.217 #3 cont.)

substituting into ⊛

$$(-64A\sin 8t - 64B\cos 8t) + 16(8A\cos 8t - 8B\sin 8t) + 64(A\sin 8t + B\cos 8t) = 32\cos 8t$$

$$-128B\sin 8t + 128A\cos 8t = 32\cos 8t$$

$$-128B = 0 \rightarrow B = 0$$

$$128A = 32 \rightarrow A = 1/4$$

$$x_p = 1/4 \sin 8t$$

$$x = x_c + x_p = c_1 e^{-8t} + c_2 t e^{-8t} + 1/4 \sin 8t$$

applying the initial conditions $x(0)=0$, $x'(0)=0$

$$0 = c_1$$

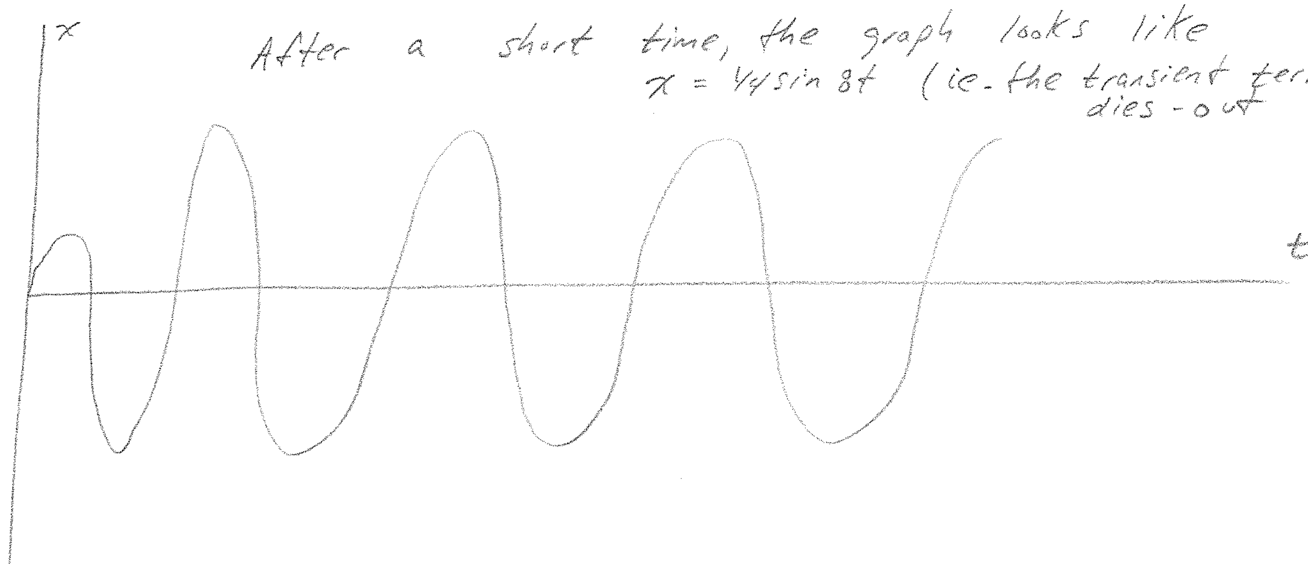
$$x = c_2 t e^{-8t} + 1/4 \sin 8t$$

$$x' = c_2 e^{-8t} - 8c_2 t e^{-8t} + 2\cos 8t$$

$$0 = c_2 + 2 \rightarrow c_2 = -2$$

$$x = \underbrace{-2te^{-8t}}_{\text{transient term}} + \underbrace{1/4 \sin 8t}_{\text{steady-state term}}$$

After a short time, the graph looks like $x = 1/4 \sin 8t$ (ie. the transient term dies-out)



p. 218
#5

wt. 6 lb.

$F = ks$

$6 = k\left(\frac{4}{12} \text{ ft}\right)$

$k = 18 \text{ lb/ft}$

damping force $3x'$

at $t=0$, $x=0$, $v=0$

$F(t) = 27 \sin 4t - 3 \cos 4t$

$$mx'' + ax' + kx = F(t)$$

$$\frac{6}{32} x'' + 3x' + 18x = 27 \sin 4t - 3\cos 4t$$

$$\otimes \quad x'' + 16x' + 96x = 144 \sin 4t - 16 \cos 4t$$

The corresponding homog. eqn.

$x'' + 16x' + 96 = 0$ has charact. eqn.

$$m^2 + 16m + 96 = 0$$

$$m = \frac{-16 \pm \sqrt{(16)^2 - 4(1)(96)}}{2(1)} = \frac{-16 \pm \sqrt{-128}}{2}$$

$$= \frac{-16 \pm 8\sqrt{2}\, i}{2} = -8 \pm 4\sqrt{2}\, i$$

The complementary soln. is

$$x_c = e^{-8t}(c_1 \sin 4\sqrt{2}t + c_2 \cos 4\sqrt{2}t)$$

$$= c_1 e^{-8t} \sin 4\sqrt{2}t + c_2 e^{-8t} \cos 4\sqrt{2}t$$

Using the method of undetermined coeffs.

The nonhomog. term $144 \sin 4t - 16 \cos 4t$ has UC sets

$S_1 = \{\sin 4t, \cos 4t\}$

$S_2 = \{\sin 4t, \cos 4t\}$

due to the overlap, we can discard S_2.

None of the elts. of S_1 are solns. of the homog. eqn.,

so we need not modify S_1

$\left(\begin{smallmatrix} p.218 \\ \#5 \end{smallmatrix} \text{ cont.} \right)$

$$x_p = A \sin 4t + B\cos 4t$$

$$x_p' = 4A \cos 4t - 4B \sin 4t$$

$$x_p'' = -16A \sin 4t - 16B\cos 4t$$

substituting into $\otimes$

$$(-16A \sin 4t - 16B\cos 4t) + 16(4A\cos 4t - 4B \sin 4t) + 96(A \sin 4t + B\cos 4t)$$

$$= 144 \sin 4t - 16 \cos 4t$$

$$(80A - 64B)\sin 4t + (80B + 64A)\cos 4t = 144 \sin 4t - 16\cos 4t$$

$$\text{(i)} \quad 80A - 64B = 144 \longrightarrow A = \frac{144 + 64B}{80} = \frac{9+4B}{5}$$

$$\text{(ii)} \quad 80B + 64A = -16$$

$$\text{(ii)} \quad 80B + 64 \cdot \left(\frac{9+4B}{5}\right) = -16$$

$$400B + 576 + 256B = -80$$

$$656B = -656 \longrightarrow B = -1$$

$$A = \frac{9 + 4(-1)}{5} = 1$$

$$x_p = \sin 4t - \cos 4t$$

$$x = x_c + x_p = C_1 e^{-8t}\sin 4\sqrt{2}\, t + C_2 e^{-8t}\cos 4\sqrt{2}\, t + \sin 4t - \cos 4t$$

Applying the initial conditions $x(0) = 0$, $x'(0) = 0$

$$0 = C_2 - 1 \longrightarrow C_2 = 1$$

$$x' = 4\sqrt{2}\, C_1 e^{-8t}\cos 4\sqrt{2}\, t - 8C_1 e^{-8t}\sin 4\sqrt{2}\, t - 4\sqrt{2}\, C_2 e^{-8t}\sin 4\sqrt{2}\, t$$
$$- 8C_2 e^{-8t}\cos 4\sqrt{2}\, t + 4\cos 4t + 4\sin 4t$$

$$0 = 4\sqrt{2}\, C_1 - 8C_2 + 4$$

$$4 = 4\sqrt{2}\, C_1$$

$$C_1 = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right) = \frac{\sqrt{2}}{2}$$

(p. 218 #5 cont.)

$$x = \frac{\sqrt{2}}{2} e^{-8t} \sin 4\sqrt{2}\, t + e^{-8t} \cos 4\sqrt{2}\, t + \sin 4t - \cos 4t$$

---

optional: we can combine terms using our reduction identity $a \sin\theta + b\cos\theta = \sqrt{a^2+b^2}\cos(\theta+\phi)$

where ϕ defined by

$$\frac{a}{\sqrt{a^2+b^2}} = -\sin\phi \quad \text{and} \quad \frac{b}{\sqrt{a^2+b^2}} = \cos\phi$$

$$x = e^{-8t}\left(\sqrt{\left(\tfrac{\sqrt{2}}{2}\right)^2 + (1)^2}\, \cos(4\sqrt{2}t + \phi_1)\right) + \sqrt{(1)^2+(-1)^2}\, \cos(4t + \phi_2)$$

ϕ_1 defined by

$$\left.\begin{aligned} \frac{\frac{\sqrt{2}}{2}}{\sqrt{\left(\frac{\sqrt{2}}{2}\right)^2+(1)^2}} &= -\sin\phi_1 \\ \frac{1}{\sqrt{\left(\frac{\sqrt{2}}{2}\right)^2+(1)^2}} &= \cos\phi_1 \end{aligned}\right\}$$

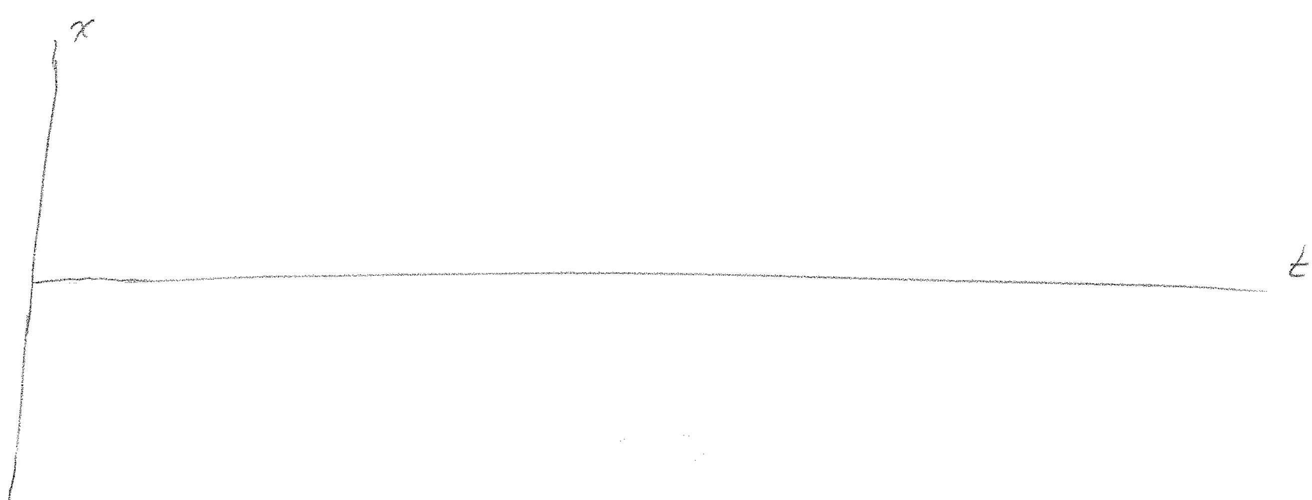
ϕ_2 defined by

$$\left.\begin{aligned} \frac{1}{\sqrt{(1)^2+(-1)^2}} &= -\sin\phi_2 \\ \frac{-1}{\sqrt{(1)^2+(-1)^2}} &= \cos\phi_2 \end{aligned}\right\}$$

$$\left.\begin{aligned} \sin\phi_1 &= -.577 \\ \cos\phi_1 &= .816 \end{aligned}\right\} \phi_1 = -.615 \quad \text{(Quadr. IV)}$$

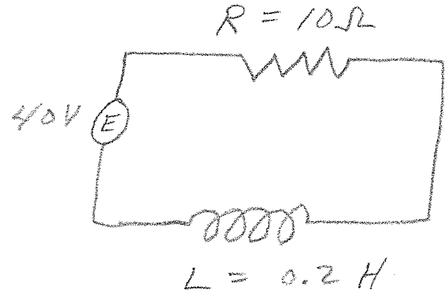
$$\left.\begin{aligned} \sin\phi_2 &= -.707 \\ \cos\phi_2 &= -.707 \end{aligned}\right\} \text{(Quadr. III)} \quad 5\pi/4 \approx 3.927$$

$$x = \underbrace{e^{-8t}\left(1.22\cos(4\sqrt{2}t - .615)\right)}_{\text{transient term}} + \underbrace{1.41\cos(4t + 3.927)}_{\text{steady-state term}}$$



p. 232
#1

at $t=0, i=0$



$$Li' + Ri + \cancel{\frac{1}{C}q} = E$$

$$.2i' + 10i = 40$$

$$i' + 50i = 200$$

1st order Linear Eqn.

integrating factor $\mu = e^{\int 50\,dt} = e^{50t}$

applying μ: $e^{50t}i' + e^{50t}(50i) = 200e^{50t}$

$$[e^{50t}i]' = [4e^{50t}]' + [C]'$$

$$e^{50t}i = 4e^{50t} + C$$

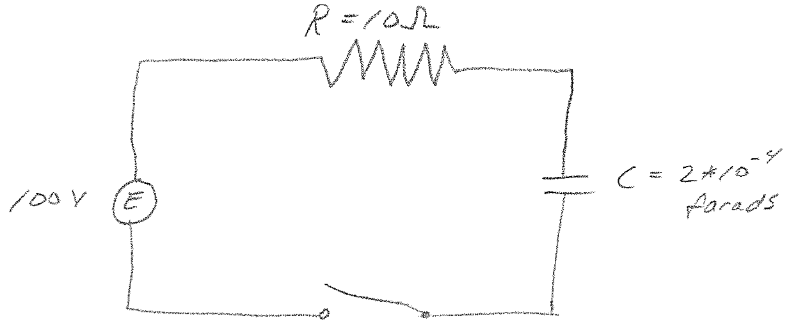
$$i = 4 + Ce^{-50t}$$

applying the initial conditions $i(0) = 0$

$$0 = 4 + C \rightarrow C = -4$$

$$i = \underbrace{4}_{\text{steady-state term}} - \underbrace{4e^{-50t}}_{\text{transient term}}$$

at $t=0,\ q=0$
$i=0$



p. 232
#3

$$\cancel{L q''}_{\to 0} + Rq' + \frac{1}{C} q = E$$

$$10 q' + \frac{1}{2*10^{-4}} q = 100$$

$$q' + 500 q = 10$$

1st order linear
int. factor $\mu = e^{\int 500 dt} = e^{500t}$

applying μ: $e^{500t} q' + 500 q e^{500t} = 10 e^{500t}$

$$\left(e^{500t} q\right)' = \left[\frac{1}{50} e^{500t}\right]' + [C]'$$

$$e^{500t} q = \frac{1}{50} e^{500t} + C$$

$$q = \frac{1}{50} + C e^{-500t}$$

applying the initial conditions $q(0) = 0$

$$0 = \frac{1}{50} + C \quad \to \quad C = -\frac{1}{50}$$

$$q = \frac{1}{50} - \frac{1}{50} e^{-500t}$$

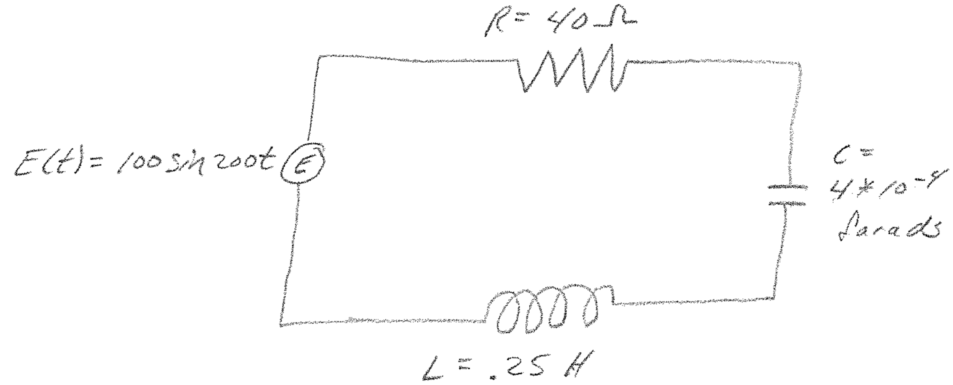
Since $i = q'$:

$$i = \underbrace{10 e^{-500t}}_{\text{transient term}}$$

(no steady-state term)

p. 232
#5

at $t=0$,
$i=0$
$q=.01$ coulombs



$$Lq'' + Rq' + \frac{1}{C}q = E$$

$$.25q'' + 40q' + \frac{1}{4 * 10^{-4}}q = 100 \sin 200t$$

$$q'' + 160q' + 10\,000q = 400 \sin 200t$$

2nd order diff. eq.

The corresponding homog. eqn.

$q'' + 160q' + 10\,000 = 0$ has charact. eqn.

$$m^2 + 160m + 10\,000 = 0$$

$$m = \frac{-160 \pm \sqrt{(160)^2 - 4(1)(10\,000)}}{2(1)} = \frac{-160 \pm \sqrt{-14400}}{2} = \frac{-160 \pm 120i}{2}$$

$$= -80 \pm 60i$$

complementary soln.

$$q_c = e^{-80t}(c_1 \sin 60t + c_2 \cos 60t)$$

method of undet. coeffs.

$$S_1 = \{\sin 200t, \cos 200t\}$$

$$q_p = A \sin 200t + B \cos 200t$$

$$q_p' = 200A \cos 200t - 200B \sin 200t$$

$$q_p'' = -40{,}000A \sin 200t - 40\,000B \cos 200t$$

(p.232 #5 cont.)

⊗ becomes

$$(-40\,000A\sin 200t - 40\,000B\cos 200t) + 160(200A\cos 200t - 200B\sin 200t)$$
$$+ 10\,000(A\sin 200t + B\cos 200t) = 400\sin 200t$$

$$(-30\,000A - 32\,000B)\sin 200t + (-30\,000B + 32\,000A)\cos 200t = 400\sin 200$$

(i) $-30\,000A - 32\,000B = 400$

(ii) $-30000B + 32\,000A = 0 \longrightarrow A = \frac{30\,000}{32\,000}B = \frac{15B}{16}$

$$\text{(i)}\ -30\,000\left(\frac{15B}{16}\right) - 32\,000B = 400$$
$$-60\,125B = 400$$
$$B = \frac{-400}{60\,125} = \frac{-16}{2405}$$

$$A = \frac{15}{16}\left(\frac{-16}{2405}\right) = \frac{-3}{481}$$

$$q_p = \frac{-3}{481}\sin 200t - \frac{16}{2405}\cos 200t$$

$$q = q_c + q_p = C_1e^{-80t}\sin 60t + C_2e^{-80t}\cos 60t - \frac{3}{481}\sin 200t$$
$$- \frac{16}{2405}\cos 200t$$

applying the initial conditions $q(0) = .01$, $q'(0) = 0$

$$.01 = C_2 - \frac{16}{2405} \rightarrow C_2 = .0167$$

$$q' = 60C_1e^{-80t}\cos 60t - 80C_1e^{-80t}\sin 60t - 60C_2e^{-80t}\sin 60t$$
$$-80C_2e^{-80t}\cos 60t - \frac{600}{481}\cos 200t + \frac{3200}{2405}\sin 200t$$

$$0 = 60C_1 - 80C_2 - \frac{600}{481}$$
$$C_1 = \frac{80(.0167) + \frac{600}{481}}{60} = .0431$$

(p.232 #5 cont.)

$$q = .0431e^{-80t}\sin 60t + .0167e^{-80t}\cos 60t - .00624\sin 200t - .00665\cos 200t$$

$$i = q' = 2.586e^{-80t}\cos 60t - 3.448e^{-80t}\sin 60t - 1.002e^{-80t}\sin 60t - 1.336e^{-80t}\cos 60t - 1.248\cos 200t + 1.33\sin 200t$$

$$= \underbrace{1.25e^{-80t}\cos 60t - 4.45e^{-80t}\sin 60t}_{\text{transient term}} \underbrace{-1.248\cos 200t + 1.33\sin 200t}_{\text{steady-state term}}$$

optional : use the trig reduction identity

$$i = \sqrt{(1.25)^2 + (-4.45)^2}\, e^{-80t}\cos(60t + \phi_1) + \sqrt{(-1.248)^2 + (1.33)^2}\cos(200t + \phi_2)$$

$$\doteq 4.62e^{-80t}\cos(60t + \phi_1) + 1.82\cos(200t + \phi_2)$$

where ϕ_1 and ϕ_2 are defined by:

$$\left.\begin{aligned} \frac{-4.45}{4.62} &= -\sin\phi_1 \\ \frac{1.25}{4.62} &= \cos\phi_1 \end{aligned}\right\} \qquad \left.\begin{aligned} \frac{1.33}{1.82} &= -\sin\phi_2 \\ \frac{-1.248}{1.82} &= \cos\phi_2 \end{aligned}\right\}$$

$$\left.\begin{aligned} \sin\phi_1 &= .963 \\ \cos\phi_1 &= .271 \end{aligned}\right\} \text{Quadr. I} \quad \phi_1 = 1.296 \text{ (radians)}$$

$$\left.\begin{aligned} \sin\phi_2 &= -.731 \\ \cos\phi_2 &= -.686 \end{aligned}\right\} \text{Quadr. III} \quad \phi_2 = .815 + \pi = 3.96 \text{ (radians)}$$

$$i = \underbrace{4.62e^{-80t}\cos(60t + 1.296)}_{\text{transient term}} + \underbrace{1.82\cos(200t + 3.96)}_{\text{steady-state term}}$$

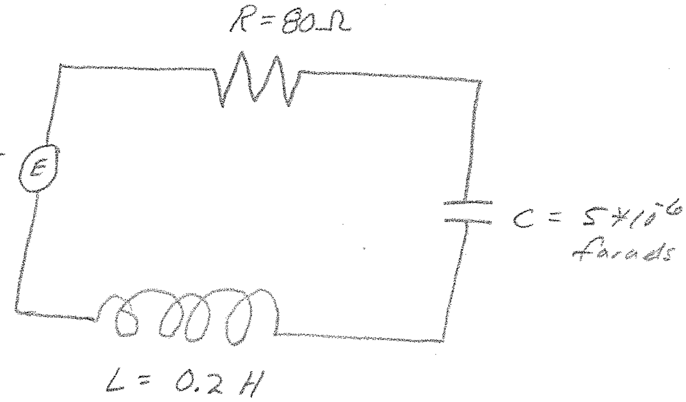
p.232
#6

at $t=0$,

$i=0$

$q=0$

$E(t)=200e^{-100t}$ V



$$Lq''+Rq'+\frac{1}{C}q=E$$

$$0.2q''+80q'+\frac{1}{5\times10^{-6}}q=200e^{-100t}$$

$$\circledast \quad q''+400q'+1\,000\,000q=1000e^{-100t}$$

The corresponding homog. eqn. has characteristic eqn.

$$m^2+400m+1\,000\,000=0$$

$$m=\frac{-400\pm\sqrt{(400)^2-4(1)(1\,000\,000)}}{2(1)}=\frac{-400\pm\sqrt{-3\,840\,000}}{2}$$

$$m=-200\pm i\,400\sqrt{6}$$

Complementary soln.

$$q_c=e^{-200t}\left(C_1\sin 400\sqrt{6}\,t+C_2\cos 400\sqrt{6}\,t\right)$$

Method of undet. coeff's.

$$S_1=\{e^{-100t}\}$$

$$q_p=Ae^{-100t}$$

$$q_p'=-100Ae^{-100t}$$

$$q_p''=10\,000Ae^{-100t}$$

$\circledast$ becomes:

$$10000Ae^{-100t}+400\left(-100Ae^{-100t}\right)+1\,000\,000Ae^{-100t}=1000e^{-100t}$$

$$970\,000Ae^{-100t}=1\,000e^{-100t}$$

$$A=\frac{1\,000}{970\,000}\approx 0.00103$$

$$q_p=.00103e^{-100t}$$

so

$$q = q_c + q_p$$

$$q = C_1 e^{-200t}\sin 400\sqrt{6}\,t + C_2 e^{-200t}\cos 400\sqrt{6}\,t + .00103 e^{-100t}$$

applying the initial conditions $q(0)=0$, $q'(0)=0$

$$0 = C_2 + .00103 \quad \rightarrow \quad C_2 = -.00103$$

$$q = C_1 e^{-200t}\sin 400\sqrt{6}\,t - .00103 e^{-200t}\cos 400\sqrt{6}\,t + .00103 e^{-100t}$$

$$q' = -200C_1 e^{-200t}\sin 400\sqrt{6}\,t + C_1(400\sqrt{6}) e^{-200t}\cos 400\sqrt{6}\,t + .206 e^{-200t}\cos 400\sqrt{6}\,t$$
$$+ 1.009 e^{-200t}\sin 400\sqrt{6}\,t - .103 e^{-100t}$$

$$0 = 400\sqrt{6}\, C_1 + .206 - .103 \quad \rightarrow \quad C_1 \approx -.000105$$

$$q = \underbrace{-.000105 e^{-200t}\sin 400\sqrt{6}\,t - .00103 e^{-200t}\cos 400\sqrt{6}\,t + .00103 e^{-100t}}_{\text{No steady-state term due to the transient nature of the forcing function } 200 e^{-100t}\ V}$$

<u>optional:</u> using the trig reduction identity

$$q = e^{-200t}\left[\sqrt{(-.000105)^2 + (-.00103)^2}\ \cos(400\sqrt{6}\,t + \phi)\right] + .00103 e^{-100t}$$

$$= e^{-200t}\left[.00104 \cos(400\sqrt{6}\,t + \phi)\right] + .00103 e^{-100t}$$

where ϕ defined by

$$\frac{-.000105}{.00104} = -\sin\phi \quad \text{and} \quad \frac{-.00103}{.00104} = \cos\phi$$

$$\underbrace{\sin\phi = .10096 \qquad \cos\phi = -.9904}_{\phi \text{ in Quadr. II} \qquad \phi = 3.0 \text{ (rad)}}$$

$$q = .00104 e^{-200t}\cos(400\sqrt{6}\,t + 3.0) + .00103 e^{-100t}$$

(p. 232 #6 cont.)

Finally,

$$i = q' = -.208\,e^{-200t}\cos(400\sqrt{6}\,t + 3.0) - 1.02\,e^{-200t}\sin(400\sqrt{6}\,t + 3.0) - .103\,e^{-100t}$$