Differential Equations Homework

p.5 1-9 odds only

p.11 1, 2, 3, 5 #5 solve 3rd degree poly. eqn. by looking for rational zeros.

p.21 1, 2, 3, 5, 6 #2(a) $y = (x^2+2)e^{-x}$
(b) $y = (x^2+3)e^{-x}$

p.36 1-17 odds only

p.46 1, 5, 6, 7, 9, 11, 15, 18, 19

p.56 1, 3, 5, 7, 9, 15, 17, 21

p.67 1, 4, 5, 7, 13

p.88 1, 3, 7, 11, 17

p.102 1, 5, 7, 9, 23, 25 #1 answer wrong (should be 35%)

p.123 7, 9

p.132 1, 5, 11

p.143 1, 5, 7, 9, 17, 19, 21, 27, 31, 37, 45

p.159 1, 3, 19, 31, 35, 39, 51, 53, 55

p.169 1, 3, 7, 13, 17

p.197 1, 5

p.208 1, 3, 4, 7, 9

p.217 1, 3, 5 (identify steady-state & transient terms)

p.232 1, 3, 5, 6 (identify steady-state & transient terms)

p.434 1, 3, 5 (Part (a) Taylor Series method only)

p.447 1, 7

p.454 1, 7

p.462 1

Test 3

(Taylor's method only)

$$y' = x + y \qquad y(0) = 1$$

p.434
#1

we seek a soln. of the form

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \cdots = \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!}x^n$$

$$y(0) = 1$$

$$y'(0) = y'(0,1) = 0 + 1 = 1$$

$$y'' = 1 + y'$$

$$y''(0) = y''(0,1) = 1 + 1 = 2$$

$$y''' = y''$$

$$y'''(0) = y'''(0,1) = 2$$

$$y^{(iv)} = y'''$$

$$y^{(iv)}(0) = y^{(iv)}(0,1) = 2$$

$$\vdots$$

$$y^{(n)}(0) = 2$$

$$y(x) = 1 + (1)x + \frac{2}{2!}x^2 + \frac{2}{3!}x^3 + \cdots + \frac{2}{n!}x^n + \cdots$$

$$= 1 + x + x^2 + \tfrac{1}{3}x^3 + \cdots + \frac{2}{n!}x^n$$

(Taylor's method only)

$$y' = 1 + xy^2 \qquad y(0) = 2$$

p. 434
#3

we seek a soln. of the form

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \cdots = \sum_{n=0}^{+\infty} \frac{y^{(n)}(0)}{n!} x^n$$

$y(0) = 2$

$y'(0) = y'(0,2) = 1 + 0(2)^2 = 1$

$$y'' = y^2 + 2xyy'$$
$$= y^2 + 2xy(1+xy^2)$$
$$= y^2 + 2xy + 2x^2y^3$$

$y''(0) = y''(0,2) = (2)^2 + 2(0)(2) + 2(0)^2(2)^3 = 4$

$$y''' = 2yy' + 2y + 2xy' + 4xy^3 + 6x^2y^2y'$$
$$= 2y(1+xy^2) + 2y + 2x(1+xy^2) + 4xy^3 + 6x^2y^2(1+xy$$
$$= 2y + 2xy^3 + 2y + 2x + 2x^2y^2 + 4xy^3 + 6x^2y^2 + 6x^3y^4$$
$$= 4y + 6xy^3 + 2x + 8x^2y^2 + 6x^3y^4$$

$y'''(0) = y'''(0,2)$
$= 4(2) + 6(0)(2)^3 + 2(0) + 8(0)^2(2)^2 + 6(0)^3(2)^4 = 8$

$$y^{(iv)} = 4y' + 6y^3 + 18xy^2y' + 2 + 16xy^2 + 16x^2yy' + 18x^2y^4 + 24x^3y^3y'$$
$$= 4(1+xy^2) + 6y^3 + 18xy^2(1+xy^2) + 2 + 16xy^2 + 16x^2y(1+xy^2) + 18x^2y^4 + 24x^3y^3(1+xy^2)$$
$$= 6 + 38xy^2 + 6y^3 + 36x^2y^4 + 16x^2y + 40x^3y^3 + 18x^2y^4 + 24x^4y^5$$

$y^{(iv)}(0) = y^{(iv)}(0,2) = 6 + 6(2)^3 = 6 + 48 = 54$

(p. 434 #3 cont.)

So the first 5 terms:

$$y(x) = 2 + (1)x + \frac{4}{2!}x^2 + \frac{8}{3!}x^3 + \frac{54}{4!}x^4 + \cdots$$

$$= 2 + x + 2x^2 + \frac{4}{3}x^3 + \frac{9}{4}x^4 + \cdots$$

(No obvious algebraic form for the closed-form solution)

(Taylor's Method only)

p.434 #4 $\quad y' = x^3 + y^3 \qquad y(0) = 3$

we need to find the coefficients for the expression

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \dots = \sum_{n=0}^{+\infty} \frac{y^{(n)}(0)}{n!}x^n$$

i.e. we need to find $\left\{ \begin{array}{l} y(0) \\ y'(0) \\ y''(0) \\ \vdots \end{array} \right.$

$y(0) = 3$ (this was given)

$y'(0) = y'(0,3)$

i.e. we need to find y' when $x=0$; but we know $y(0)=3$, so what we need then is to find y' when $x=0$ and $y=3$

$$y'(0) = y'(0,3) = (0)^3 + (3)^3 = 27$$

$$y'' = 3x^2 + 3y^2(y')$$
$$= 3x^2 + 3y^2(x^3+y^3) = 3x^2 + 3x^3y^2 + 3y^5$$

$$y''(0) = y''(0,3) = 3(0)^2 + 3(0)^3(3)^2 + 3(3)^5$$
$$= 729$$

$$y''' = 6x + 9x^2y^2 + 6x^3y(y') + 15y^4(y')$$
$$= 6x + 9x^2y^2 + 6x^3y(x^3+y^3) + 15y^4(x^3+y^3)$$
$$= 6x + 9x^2y^2 + 6x^6y + 6x^3y^4 + 15x^3y^4 + 15y^7$$

$$y'''(0) = y'''(0,3) = 15(3)^7 = 32805$$

(p. 434 #4 cont.)

$$y^{(iv)} = 6 + 18xy^2 + 18x^2y(y') + 36x^5y + 6x^6(y')$$
$$+ 18x^2y^4 + 24x^3y^3(y') + 45x^2y^4 + 60x^3y^3(y')$$
$$+ 105y^6(y')$$

$$y^{(iv)}(0) = y^{(iv)}(0,3) = 6 + 105(3)^6(27) = 2\ 066\ 721$$

It appears as though the most significant term in each case is the final term.
Let's track the progress of this final term.

derivative	final term
y'	y^3
y''	$3y^5$
y'''	$15y^7$
$y^{(iv)}$	$105y^9$

(Taylor's method only)

p. 434
#5

$$y' = x + \sin y \qquad y(0) = 0$$

we seek a soln of the form

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \dots = \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} x^n$$

$$y(0) = 0$$
$$y'(0) = y'(0,0) = 0 + \sin(0) = 0$$

$$y'' = 1 + \cos y\,(y')$$
$$= 1 + \cos y\,(x + \sin y)$$
$$= 1 + x\cos y + \sin y \cos y$$

$$y''(0) = y''(0,0) = 1 + 0\cos 0 + \sin 0 \cos 0 = 1$$

$$y''' = \cos y - x\sin y\,(y') - \sin^2 y\,(y') + \cos^2 y\,(y')$$
$$= \cos y - x\sin y\,(x + \sin y) - \sin^2 y\,(x + \sin y) + \cos^2 y\,(x + \sin y)$$
$$= \cos y - x^2 \sin y - x\sin^2 y - x\sin^2 y - \sin^3 y + x\cos^2 y + \sin y \cos^2 y$$

$$= \cos y - x^2 \sin y - 2x\sin^2 y - \sin^3 y + x\cos^2 y + \sin y \cos^2 y$$

$$y'''(0) = y'''(0,0) = \cos 0 = 1$$

$$y^{(iv)} = -\sin y\,(y') - 2x\sin y - x^2\cos y\,(y') - 2\sin^2 y - 4x\sin y\cos y\,(y')$$
$$- 3\sin^2 y\cos y\,(y') + \cos^2 y - 2x\cos y\sin y\,(y') + \cos^3 y\,(y')$$
$$- 2\sin y\cos y\,(-\sin y)(y')$$

$$= -\sin y\,(x + \sin y) - 2x\sin y - x^2\cos y\,(x + \sin y) - 2\sin^2 y$$
$$- 4x\sin y\cos y\,(x + \sin y) - 3\sin^2 y\cos y\,(x + \sin y) + \cos^2 y$$
$$- 2x\cos y\sin y\,(x + \sin y) + \cos^3 y\,(x + \sin y)$$
$$+ 2\sin^2 y\cos y\,(x + \sin y)$$

(p. 434 #5 cont.)

$$y^{(iv)}(0) = y^{(iv)}(0,0) = \cos^2(0) = 1$$

so the 1st 3 terms are given by

$$y(x) = 0 + (0)x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$$

$$= \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots$$

Euler Method

p. 447 #1 $\quad y' = x - 2y \quad , \quad y(0) = 1 \quad , \quad h = .2$

$$f(x,y) = x - 2y$$

$x_0 = 0 \quad , \quad x_1 = .2 \quad , \quad x_2 = .4 \, , \, x_3 = .6 \quad , \quad x_4 = .8 \, , \, x_5 = 1.0$

$y_0 = 1$

$$y_1 = y_0 + h\,f(x_0, y_0)$$
$$= 1 + .2\,f(0,1)$$
$$= 1 + .2\,(0 - 2(1)) = 1 - .4 = .6$$

$$y_2 = y_1 + h\,f(x_1, y_1)$$
$$= .6 + .2\,(.2 - 2(.6)) = .6 - .2 = .4$$

$$y_3 = y_2 + h\,f(x_2, y_2)$$
$$= .4 + .2\,(.4 - 2(.4)) = .32$$

$$y_4 = y_3 + h\,f(x_3, y_3)$$
$$= .32 + .2\,(.6 - 2(.32)) = .312$$

$$y_5 = y_4 + h\,f(x_4, y_4)$$
$$= .312 + .2\,(.8 - 2(.312)) = \underline{.3472}$$

(p. 447 #1 cont.)

Exact solution:

$y' + 2y = x$ Linear

$\mu = e^{\int 2\,dx} = e^{2x}$

applying μ

$e^{2x}y' + 2ye^{2x} = xe^{2x} \longrightarrow$

parts: $u = x \quad dv = e^{2x}dx$, $du = dx \quad v = \frac{1}{2}e^{2x}$

$\int xe^{2x}\,dx$
$= x(\frac{1}{2}e^{2x}) - \int \frac{1}{2}e^{2x}\,dx$
$= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}$

$(ye^{2x})' = (\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x})' + (c)'$

$ye^{2x} = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c \rightarrow \boxed{y = \frac{1}{2}x - \frac{1}{4} + ce^{-2x}}$

applying initial conditions $y(0) = 1$

$1 = -\frac{1}{4} + c \quad \rightarrow \quad c = \frac{5}{4}$

$\phi(x) = y = \frac{1}{2}x - \frac{1}{4} + \frac{5}{4}e^{-2x}$

$\phi(.2) = \frac{1}{2}(.2) - \frac{1}{4} + \frac{5}{4}e^{-2(.2)} \approx .6879001$

$\phi(.4) = \dots$

x_n	Exact solution	Euler approx.	Error	% rel. error
0	1	1	0	0
.2	.6879001	.6	.0879001	$100\left(\frac{.0879001}{.6879001}\right) = 12.77803\%$
.4	.5116612	.4	.1116612	21.82327 %
.6	.4264928	.32	.1064928	24.96943 %
.8	.4023706	.312	.0903706	22.45954 %
1.0	.4191691	.3472	.0719691	17.16947 %

$$y' = \sin 2x + y \quad , \quad y(0) = 1 \quad , \quad h = .25$$

p. 448
#7

$$f(x,y) = \sin 2x + y$$

$x_0 = 0$, $x_1 = .25$, $x_2 = .50$, $x_3 = .75$, $x_4 = 1.0$, $x_5 = 1.25$,
$x_6 = 1.50$, $x_7 = 1.75$, $x_8 = 2.0$

$y_0 = 1$

$y_1 = y_0 + h f(x_0, y_0)$
$= 1 + .25(\sin 0 + 1) = 1.25$

$y_2 = y_1 + h f(x_1, y_1)$
$= 1.25 + .25(\sin .50 + 1.25) = 1.682356385$

$y_3 = y_2 + h f(x_2, y_2)$
$= 1.682356385 + .25(\sin 1.00 + 1.682356385) = 2.313313227$

$y_4 = y_3 + h f(x_3, y_3)$
$= 2.313313227 + .25(\sin 1.50 + 2.313313227) = 3.141015281$

$y_5 = y_4 + h f(x_4, y_4)$
$= 3.141015281 + .25(\sin 2.00 + 3.141015281) = 4.153593457$

$y_6 = y_5 + h f(x_5, y_5)$
$= 4.153593457 + .25(\sin 2.50 + 4.153593457) = 5.341609858$

$y_7 = y_6 + h f(x_6, y_6)$
$= 5.341609858 + .25(\sin 3.00 + 5.341609858) = 6.712292324$

$y_8 = y_7 + h f(x_7, y_7)$
$= 6.712292324 + .25(\sin 3.50 + 6.712292324) = 8.302669598$

(p. 44: #7 cont.)

<u>Exact soln:</u>

$y' - y = \sin 2x$ <u>Linear</u>

$\mu = e^{\int -dx} = e^{-x}$

applying μ:

parts twice - see next pg.

$e^{-x}y' - e^{-x}y = e^{-x}\sin 2x$

$(e^{-x}y)' = \left[-\tfrac{1}{5}e^{-x}(\sin 2x + 2\cos 2x)\right]' + (c)'$

$e^{-x}y = -\tfrac{1}{5}e^{-x}(\sin 2x + 2\cos 2x) + c$

$\phi(x) = y = -\tfrac{1}{5}(\sin 2x + 2\cos 2x) + ce^{x}$

applying initial conditions $\phi(0) = 1$

$1 = -\tfrac{1}{5}(2) + c \rightarrow c = 7/5$

$\phi(x) = -\tfrac{1}{5}(\sin 2x + 2\cos 2x) + \tfrac{7}{5}e^{x}$

$\phi(.25) = -\tfrac{1}{5}(\sin(.50) + 2\cos(.50)) + \tfrac{7}{5}e^{.25} = 1.3507174$

$\phi(.50) = \ldots$

x_n	Exact soln.	Euler approx.	Error	% rel. error
0	1			
.25	1.3507174	1.25	0.1007174	7.45659%
.50				
.75				
1.00				
1.25				
1.50				
1.75				
2.00	10.757496	8.302669598	2.454826	22.81968%

(p. 448 #7 cont.)

Here is the integration from the previous page

$$I = \int e^{-x} \sin 2x \, dx$$

<u>parts:</u>

$u = \sin 2x \qquad dv = e^{-x}\,dx$

$du = 2\cos 2x \, dx \qquad v = -e^{-x}$

$(uv - \int v\,du)$:

$$I = (\sin 2x)(-e^{-x}) - \int (-e^{-x})(2\cos 2x \, dx)$$

$$I = -e^{-x}\sin 2x + 2\int e^{-x}\cos 2x \, dx$$

<u>parts again:</u>

$u = \cos 2x \qquad dv = e^{-x}\,dx$

$du = -2\sin 2x \, dx \qquad v = -e^{-x}$

$$I = -e^{-x}\sin 2x + 2\left[(\cos 2x)(-e^{-x}) - \int (-e^{-x})(-2\sin 2x \, dx)\right]$$

$$I = -e^{-x}\sin 2x - 2e^{-x}\cos 2x \underbrace{- 4\underbrace{\int e^{-x}\sin 2x \, dx}_{I}}_{4I}$$

$$5I = -e^{-x}(\sin 2x + 2\cos 2x)$$

$$I = -\tfrac{1}{5} e^{-x}(\sin 2x + 2\cos 2x)$$

$$y' = x - 2y \quad , \quad y(0) = 1 \quad , \quad h = .2$$

p. 454 $\qquad f(x,y) = x - 2y$

#1 $\qquad x_0 = 0,\ y_0 = 1$

$$x_0 = 0 \ , \ x_1 = .2 \ , \ x_2 = .4 \ , \ x_3 = .6 \ , \ x_4 = .8 \ , \ x_5 = 1.0$$

$$\hat{y}_1 = y_0 + h f(x_0, y_0) = 1 + .2\left(0 - 2(1)\right) = .6$$

$$y_1 = y_0 + h\left[\frac{f(x_0,y_0) + f(x_1,\hat{y}_1)}{2}\right]$$

$$= 1 + .2\left[\frac{(0 - 2(1)) + (.2 - 2(.6))}{2}\right] = 1 + .2\left[\frac{-2 + (-1)}{2}\right] = .7$$

$$\hat{y}_2 = y_1 + h f(x_1, y_1) = .7 + .2\left(.2 - 2(.7)\right) = .46$$

$$y_2 = y_1 + h\left[\frac{f(x_1,y_1) + f(x_2,\hat{y}_2)}{2}\right]$$

$$= .7 + .2\left[\frac{(.2 - 2(.7)) + (.4 - 2(.46))}{2}\right] = .7 + .2\left[\frac{-1.2 + (-.52)}{2}\right] = .528$$

$$\hat{y}_3 = y_2 + h f(x_2, y_2) = .528 + .2\left(.4 - 2(.528)\right) = .3968$$

$$y_3 = y_2 + h\left(\frac{f(x_2,y_2) + f(x_3,\hat{y}_3)}{2}\right)$$

$$= .528 + .2\left[\frac{(.4 - 2(.528)) + (.6 - 2(.3968))}{2}\right] = .528 + .2\left(\frac{-.656 + (-.1936)}{2}\right) = .44304$$

$$\hat{y}_4 = y_3 + h f(x_3, y_3) = .44304 + .2\left(.6 - 2(.44304)\right) = .385824$$

$$y_4 = y_3 + h\left[\frac{f(x_3,y_3) + f(x_4,\hat{y}_4)}{2}\right]$$

$$= .44304 + .2\left[\frac{(.6 - 2(.44304)) + (.8 - 2(.385824))}{2}\right]$$

$$= .44304 + .2\left[\frac{-.28608 + (.028352)}{2}\right] = .4172672$$

(p.454 #1 cont.)

$$\hat{y}_5 = y_4 + h f(x_4, y_4) = .4172672 + .2(.8 - 2(.4172672)) = .4103603$$

$$y_5 = y_4 + h\left[\frac{f(x_4,y_4) + f(x_5,\hat{y}_5)}{2}\right]$$

$$= .4172672 + .2\left[\frac{(.8 - 2(.4172672)) + (1.0 - 2(.4103603))}{2}\right]$$

$$= .4172672 + .2\left[\frac{-.0345344 + .1792794}{2}\right] = .4317417$$

using the exact values from p.447, #1:

x_n	Exact Solution	Improved Euler	Error	% Rel. Error
0	1	1	0	0%
.2	.6879001	.7	.0120999	1.75896%
.4	.5116612	.528	.0163388	3.19328%
.6	.4264928	.44304	.0165472	3.87983%
.8	.4023706	.4172672	.0148966	3.70221%
1.0	.4191691	.4317417	.0125726	2.99941%

$y' = \sin 2x + y \quad , \ y(0) = 1 \quad , \ h = .25$

p.455
#7

$f(x,y) = \sin 2x + y$

$x_0 = 0, \ y_0 = 1$

$x_0 = 0, \ x_1 = .25, \ x_2 = .50, \ x_3 = .75, \ x_4 = 1.0, \ x_5 = 1.25, \ x_6 = 1.5, \ x_7 = 1.75, \ x_8 =$

$\hat{y}_1 = y_0 + h f(x_0, y_0) = 1 + .25(\sin 2(0) + 1) = 1.25$

$$y_1 = y_0 + h\left[\frac{f(x_0,y_0) + f(x_1,\hat{y}_1)}{2}\right]$$

$$= 1 + .25\left[\frac{(\sin 2(0) + 1) + (\sin 2(.25) + 1.25)}{2}\right] = 1 + .25\left(\frac{1 + 1.7294255}{2}\right)$$

$$= 1.3411782$$

$\hat{y}_2 = y_1 + h f(x_1, y_1) = 1.3411782 + .25(\sin 2(.25) + 1.3411782) = 1.7963291$

$$y_2 = y_1 + h\left[\frac{f(x_1,y_1) + f(x_2,\hat{y}_2)}{2}\right]$$

$$= 1.3411782 + .25\left[\frac{(\sin 2(.25) + 1.3411782) + (\sin 2(.50) + 1.796329)}{2}\right]$$

$$= 1.3411782 + .25\left[\frac{1.8206037 + 2.6378001}{2}\right] = 1.8984787$$

$\hat{y}_3 = y_2 + h f(x_2, y_2) = 1.8984787 + .25(\sin 2(.50) + 1.8984787)$
$= 2.5834661$

$$y_3 = y_2 + h\left[\frac{f(x_2,y_2) + f(x_3,\hat{y}_3)}{2}\right]$$

$$= 1.8984787 + .25\left[\frac{(\sin 2(.50) + 1.8984787) + (\sin 2(.75) + 2.5834661)}{2}\right]$$

$$= 1.8984787 + .25\left(\frac{2.7399497 + 3.5809611}{2}\right) = 2.6885925$$

(p. 455 #7 cont.)

$$\hat{y}_4 = y_3 + h\,f(x_3, y_3) = 2.6885925 + .25(\sin 2(.75) + 2.6885925)$$
$$= 3.6101144$$

$$y_4 = y_3 + h\left[\frac{f(x_3,y_3) + f(x_4,\hat{y}_4)}{2}\right]$$

$$= 2.6885925 + .25\left[\frac{(\sin 2(.75) + 2.6885925) + (\sin 2(1.0) + 3.6101144)}{2}\right]$$

$$= 3.71428$$

$$\hat{y}_5 = y_4 + h\,f(x_4,y_4) = 3.71428 + .25(\sin 2(1.0) + 3.71428) = 4.8701743$$

$$y_5 = y_4 + h\left[\frac{f(x_4,y_4) + f(x_5,\hat{y}_5)}{2}\right]$$
$$= 3.71428 + .25\left[\frac{(\sin 2(1.0) + 3.71428) + (\sin 2(1.25) + 4.8701743)}{2}\right]$$

$$= 4.975808$$

$$\hat{y}_6 = y_5 + h\,f(x_5,y_5) = 4.975808 + .25(\sin 2(1.25) + 4.975808) =$$
$$= 6.369378$$

$$y_6 = y_5 + h\left[\frac{f(x_5,y_5) + f(x_6,\hat{y}_4)}{2}\right]$$

$$= 4.975808 + .25\left[\frac{(\sin 2(1.25) + 4.975808) + (\sin 2(1.5) + 6.369378)}{2}\right]$$

$$= 6.4864052$$

$$\hat{y}_7 = y_6 + h\,f(x_6,y_6) = 6.4864052 + .25(\sin 2(1.5) + 6.4864052)$$

$$= 8.1432865$$

(p.455 #7 cont.)

$$y_7 = y_6 + h\left[\frac{f(x_6,y_6) + f(x_7,\hat{y}_7)}{2}\right]$$

$$= 6.4864052 + .25\left[\frac{(\sin 2(1.5) + 6.4864052) + (\sin 2(1.75) + 8.1432865)}{2}\right]$$

$$= 8.2889088$$

$$\hat{y}_8 = y_7 + h\, f(x_7, y_7) = 8.2889088 + .25(\sin 2(1.75) + 8.2889088)$$

$$= 10.27344$$

$$y_8 = y_7 + h\left[\frac{f(x_7,y_7) + f(x_8,\hat{y}_8)}{2}\right]$$

$$= 8.2889088 + .25\left[\frac{(\sin 2(1.75) + 8.2889088) + (\sin 2(2.0) + 10.27344)}{2}\right]$$

$$= 10.470754$$

x_n	Exact Soln.	Improved Euler Approx.	Error	% rel. error
0	1	1	0	0%
.25	1.3507174	1.3411782		
.50		1.8984787		
.75		2.6885925		
1.0		3.71428		
1.25		4.975808		
1.50		6.4864052		
1.75		8.2889088		
2.0	10.757496	10.470754		

(Runge - Kutta Method)

p. 462 #1

$$y' = x - 2y \quad , \quad y(0) = 1 \quad , \quad h = 0.2$$

$$f(x,y) = x - 2y$$

$x_0 = 0, \quad x_1 = .2, \quad x_2 = .4, \quad x_3 = .6, \quad x_4 = .8, \quad x_5 = 1.0$

$y_0 = 1$

$$k_1 = h f(x_0, y_0) = .2(0 - 2(1)) = -.4$$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= h f\left(0 + \frac{.2}{2}, 1 + \frac{-.4}{2}\right) = h f(.1, .8)$$

$$= .2(.1 - 2(.8)) = -.3$$

$$k_3 = h f\left(x_0 + h/2, y_0 + \frac{k_2}{2}\right)$$

$$= h f\left(.1, 1 + \frac{-.3}{2}\right) = h f(.1, .85)$$

$$= .2(.1 - 2(.85)) = -.32$$

$$k_4 = h f(x_0 + h, y_0 + k_3)$$

$$= h f(.2, 1 + (-.32)) = h f(.1, .68)$$

$$= .2(.2 - 2(.68)) = -.232$$

$$K = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}((-.4) + 2(-.3) + 2(-.32) + (-.232))$$

$$= -.312$$

$$y_1 = y_0 + K = 1 + (-.312) = .688$$

(p.762 #1 cont.)

Now to find y_2:

$$k_1 = h f(x_1, y_1) = .2 f(.2, .688) = .2(.2 - 2(.688)) = -.2352$$

$$k_2 = h f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$
$$= h f\left(.2 + \frac{.2}{2}, .688 + \frac{-.2352}{2}\right) = h f(.3, .5704)$$
$$= .2(.3 - 2(.5704)) = -.16816$$

$$k_3 = h f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$
$$= h f\left(.3, .688 + \frac{-.16816}{2}\right) = h f(.3, .60392)$$
$$= .2(.3 - 2(.60392)) = -.181568$$

$$k_4 = h f(x_1 + h, y_1 + k_3)$$
$$= h f(.2 + .2, .688 + (-.181568)) = h f(.4, .506432)$$
$$= .2(.4 - 2(.506432)) = -.1225728$$

$$K = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}((-.2352) + 2(-.16816) + 2(-.181568) + (-.1225728))$$

$$= -.1762048$$

$$y_2 = y_1 + K = .688 + (-.1762048) = .5117952$$

(p. 766 #1 cont.)

Now to find y_3:

$$k_1 = hf(x_2, y_2) = hf(.4, .5117952) = .2(.4 - 2(.5117952))$$
$$= -.1247181$$

$$k_2 = hf(x_2 + \tfrac{h}{2}, y_2 + \tfrac{k_1}{2})$$
$$= hf(.4 + \tfrac{.2}{2}, .5117952 + \tfrac{-.1247181}{2}) = hf(.5, .4494362)$$
$$= .2(.5 - 2(.4494362)) = -.0797745$$

$$k_3 = hf(x_2 + h/2, y_2 + k_2/2)$$
$$= hf(.5, .5117952 + \tfrac{-.0797745}{2}) = hf(.5, .471908)$$
$$= .2(.5 - 2(.471908)) = -.0887632$$

$$k_4 = hf(x_2 + h, y_2 + k_3)$$
$$= hf(.4 + .2, .5117952 + (-.0887632)) = hf(.6, .423032)$$
$$= .2(.6 - 2(.423032)) = -.0492128$$

$$K = \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= \tfrac{1}{6}((-.1247181) + 2(-.0797745) + 2(-.0887632) + (-.0492128))$$
$$= -.0851677$$

$$y_3 = y_2 + K = .5117952 + (-.0851677) = .4266275$$

(P.462 #1 Cont.)

Now to find y_4:

$$k_1 = hf(x_3, y_3) = hf(.6, .4266275) = .2(.6 - 2(.4266275))$$
$$= -.050651$$

$$k_2 = hf(x_3 + \tfrac{h}{2}, y_3 + k_1/2)$$
$$= hf(.6 + \tfrac{.2}{2}, .4266275 + \tfrac{-.050651}{2}) = hf(.7, .401302)$$
$$= .2(.7 - 2(.401302)) = -.0205208$$

$$k_3 = hf(x_3 + h/2, y_3 + k_2/2)$$
$$= hf(.7, .4266275 + \tfrac{-.020528}{2}) = hf(.7, .4163671)$$
$$= .2(.7 - 2(.4163671)) = -.0265468$$

$$k_4 = hf(x_3 + h, y_3 + k_3)$$
$$= hf(.6 + .2, .4266275 + (-.0265468)) = hf(.8, .4000807)$$
$$= .2(.8 - 2(.4000807)) = -.0000323$$

$$K = \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= \tfrac{1}{6}((-.050651) + 2(-.0205208) + 2(-.0265468) + (-.0000323))$$
$$= -.0241364$$

$$y_4 = y_3 + K = .4266275 + (-.0241364) = .4024911$$

(p.462 #1 cont.)

Finally: to find y_5

$k_1 = hf(x_4, y_4) = hf(.8, .4024911) = .2(.8 - 2(.4024911)) = -.0009964$

$k_2 = hf(x_4 + h/2, y_4 + k_1/2)$

$= hf(.8 + \frac{.2}{2}, .4024911 + \frac{-.0009964}{2}) = hf(.9, .4019929)$

$= .2(.9 - 2(.4019929)) = .0192029$

$k_3 = hf(x_4 + h/2, y_4 + k_2/2)$

$= hf(.9, .4024911 + \frac{.0192029}{2}) = hf(.9, .4120925)$

$= .2(.9 - 2(.4120925)) = .015163$

$k_4 = hf(x_4 + h, y_4 + k_3)$

$= hf(.8 + .2, .4024911 + .015163) = hf(1.0, .4176541)$

$= .2(1.0 - 2(.4176541)) = .0329384$

$K = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= \frac{1}{6}((-.0009964) + 2(.0192029) + 2(.015163) + .0329384)$

$= .016779$

$y_5 = y_4 + K = .4024911 + .016779 = .41927$

(p. 762 #1 cont.)

Results:

x_n	Exact Soln.	Runge-Kutta	Error	% Rel. Error
0	1	1	0	0%
0.2	.6879001	.688	.0000999	0.01452 %
0.4	.5116612	.5117952	.000134	0.02619 %
0.6	.4264928	.4266275	.0001347	0.03158%
0.8	.4023706	.4024911	.0001205	0.02995%
1.0	.4191691	.41927	.0001009	0.02407 %