Differential Equations Homework

Test 4

p. 296 1, 3, 5, 7, 9 standard method

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Laplace Transform Book - All

Laplace Transform solutions of Differential Equations

A Programmed Text

by Robert D. Strum and John R. Ward

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p. 297 #1

Standard Differential Operator Method:

$$x' + y' - 2x - 4y = e^t \longrightarrow (D-2)x + (D-4)y = e^t$$
$$x' + y' - y = e^{4t} \longrightarrow Dx + (D-1)y = e^{4t}$$

Eliminate x by applying the operator $-D$ to the 1st eqn. + $(D-2)$ to the 2nd

$$-D(D-2)x - D(D-4)y = -De^t$$
$$D(D-2)x + (D-1)(D-2)y = (D-2)e^{4t}$$

$$-D(D-4)y + (D-1)(D-2)y = -De^t + (D-2)e^{4t}$$
$$(-D^2+4D)y + (D^2-3D+2)y = -De^t + (D-2)e^{4t}$$
$$(D+2)y = -e^t + 4e^{4t} - 2e^{4t}$$
$$(D+2)y = -e^t + 2e^{4t}$$

This is a 1st order linear eqn:

$$y' + 2y = -e^t + 2e^{4t}$$

$$\mu = e^{\int 2\,dt} = e^{2t} \quad \text{applying } \mu:$$

$$e^{2t}y' + 2e^{2t}y = -e^{3t} + 2e^{6t}$$
$$d(e^{2t}y) = d\left(-\tfrac{1}{3}e^{3t} + \tfrac{1}{3}e^{6t} + k\right)$$
$$e^{2t}y = -\tfrac{1}{3}e^{3t} + \tfrac{1}{3}e^{6t} + k$$
$$\boxed{y = -\tfrac{1}{3}e^t + \tfrac{1}{3}e^{4t} + ke^{-2t}}$$

Eliminate y by applying the operator $(D-1)$ to 1st eqn. & $-(D-4)$ to 2nd eqn

$$(D-1)(D-2)x + (D-1)(D-4)y = (D-1)e^t$$
$$-D(D-4)x - (D-1)(D-4)y = -(D-4)e^{4t}$$

$$(D-1)(D-2)x - D(D-4)x = (D-1)e^t - (D-4)e^{4t}$$
$$(D^2-3D+2)x + (-D^2+4D)x = e^t - e^t - 4e^{4t} + 4e^{4t}$$
$$(D+2)x = 0$$

This is a 1st order linear eqn:

$$x' + 2x = 0$$

Also separable:

$$\frac{dx}{dt} = -2x$$
$$\int \frac{dx}{x} = \int -2\,dt$$
$$\ln x = -2t + C_0$$
$$x = e^{-2t+C_0} \rightarrow \boxed{x = Ce^{-2t}}$$

please note: we could relate c & k to eliminate one of them...

- see pp. 292 + 293

p. 296 #3

Standard Differential operator method:

$$5x' + y' - 3x + y = 0 \longrightarrow (5D-3)x + (D+1)y = 0$$
$$4x' + y' - 3x = -3t \longrightarrow (4D-3)x + Dy = -3t$$

Eliminate x by applying the operator $(4D-3)$ to 1st eqn. + $-(5D-3)$ to 2nd eqn.

$$(4D-3)(5D-3)x + (4D-3)(D+1)y = 0$$
$$-(5D-3)(4D-3)x - D(5D-3)y = -(5D-3)(-3t)$$

$$(4D-3)(D+1)y - D(5D-3)y = -(5D-3)(-3t)$$
$$(4D^2+D-3)y + (-5D^2+3D)y = 15-9t$$
$$(-D^2+4D-3)y = 15-9t$$

This is a 2nd order linear eqn:
$$y'' - 4y' + 3y = 9t - 15$$

The corresp. homog. eqn.
$$y'' - 4y' + 3y = 0$$
has characteristic eqn
$$m^2 - 4m + 3 = 0$$
$$(m-1)(m-3) = 0 \rightarrow y_c = k_1 e^t + k_2 e^{3t}$$

Now we need a particular soln. of
$$y'' - 4y' + 3y = 9t - 15$$

method of undet. coeff's. $S_1 = \{t, 1\}$

$$y_p = At + B$$
$$y_p' = A$$
$$y_p'' = 0$$

$$0 - 4A + 3(At+B) = 9t - 15$$
$$3At + (3B - 4A) = 9t - 15$$

$3A = 9$ $\quad$ $3B - 4A = -15$

$A = 3$ $\quad$ $B = -1$

$$y_p = 3t - 1$$

$$\boxed{y = y_c + y_p = k_1 e^t + k_2 e^{3t} + 3t - 1}$$

Book says "+" anyone see an error?

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(p. 296 #3 continued)

Eliminate y by applying the operator $-D$ to 1st eqn. + $(D+1)$ to 2nd eqn.

$$-D(5D-3)x - D(D+1)y = 0$$
$$(D+1)(4D-3)x + D(D+1)y = (D+1)(-3t)$$

$$-D(5D-3)x + (D+1)(4D-3)x = (D+1)(-3t)$$
$$(-5D^2+3D)x + (4D^2+D-3)x = -3-3t$$
$$(-D^2+4D-3)x = -3t-3$$

This is a 2nd order linear eqn:
$$x'' - 4x' + 3x = 3t+3$$

The corresp. homog. eqn.
$$x'' - 4x' + 3x = 0$$
has charact. eqn.
$$m^2 - 4m + 3 = 0$$
$$(m-1)(m-3) = 0 \longrightarrow x_c = c_1 e^t + c_2 e^{3t}$$

Now we need a particular soln. of
$$x'' - 4x' + 3x = 3t+3$$

UC set $S_1 = \{t, 1\}$

$$x_p = At + B$$
$$x_p' = A$$
$$x_p'' = 0$$

$$\longrightarrow 0 - 4A + 3(At+B) = 3t+3$$
$$3At + (3B-4A) = 3t+3$$

$3At = 3t$ $\qquad$ $3B - 4A = 3$

$A = 1$ $\qquad$ $B = 7/3$

$$x_p = t + 7/3$$

$$\boxed{x = x_c + x_p = c_1 e^t + c_2 e^{3t} + t + 7/3}$$

please note:
we could relate c_1, c_2, k_1, k_2 + eliminate two of them...

p. 296 #5

Standard Differential Operator method:

$$x' + y' - x - 3y = e^t \longrightarrow (D-1)x + (D-3)y = e^t$$
$$x' + y' + x = e^{3t} \longrightarrow (D+1)x + Dy = e^{3t}$$

Eliminate x:
$$(D+1)(D-1)x + (D+1)(D-3)y = (D+1)e^t$$
$$-(D-1)(D+1)x - D(D-1)y = -(D-1)e^{3t}$$
$$(D^2-2D-3)y + (-D^2+D)y = e^t + e^t - 3e^{3t} + e^{3t}$$
$$(-D-3)y = 2e^t - 2e^{3t}$$

This is a 1st order linear eqn:
$$y' + 3y = 2e^{3t} - 2e^t$$
$$\mu = e^{\int 3dt} = e^{3t} \quad ; \text{ applying } \mu:$$
$$e^{3t}y' + 3e^{3t}y = 2e^{6t} - 2e^{4t}$$
$$d(e^{3t}y) = d\left(\tfrac{1}{3}e^{6t} - \tfrac{1}{2}e^{4t} + c\right)$$
$$\boxed{y = \tfrac{1}{3}e^{3t} - \tfrac{1}{2}e^t + ce^{-3t}}$$

Eliminate y:
$$-D(D-1)x - D(D-3)y = -De^t$$
$$(D-3)(D+1)x + D(D-3)y = (D-3)e^{3t}$$
$$(-D^2+D)x + (D^2-2D-3)x = -e^t + 3e^{3t} - 3e^{3t}$$
$$(-D-3)x = -e^t$$

This is a 1st order linear eqn:
$$x' + 3x = e^t$$
$$\mu = e^{\int 3dt} = e^{3t} \quad ; \text{ applying } \mu:$$
$$e^{3t}x' + 3e^{3t}x = e^{4t}$$
$$d(e^{3t}x) = d\left(\tfrac{1}{4}e^{4t} + k\right)$$
$$\boxed{x = \tfrac{1}{4}e^t + ke^{-3t}}$$

Please note: we could relate c & k and eliminate one of them

p. 296 #7

Standard Differential operator method:

$$5x' + y' - 5x - y = 0 \longrightarrow (5D-5)x + (D-1)y = 0$$
$$4x' + y' - 3x = t \longrightarrow (4D-3)x + Dy = t$$

Eliminate x:
$$(4D-3)(5D-5)x + (4D-3)(D-1)y = 0$$
$$-(5D-5)(4D-3)x - D(5D-5)y = -(5D-5)t$$

$$(4D^2-7D+3)y + (-5D^2+5D)y = -5+5t$$
$$(-D^2-2D+3)y = -5+5t$$

This is a 2nd order linear eqn:
$$y'' + 2y' - 3y = 5-5t$$

The corresp. homog. eqn.
$$y'' + 2y' - 3y = 0$$
has charact. eqn.
$$m^2 + 2m - 3 = 0$$
$$(m+3)(m-1) = 0 \rightarrow y_c = C_1e^{-3t} + C_2e^{t}$$

Now we need a particular soln. of
$$y'' + 2y' - 3y = 5-5t$$

UC set $S_1 = \{t, 1\}$

$$y_p = At + B$$
$$y_p' = A$$
$$y_p'' = 0$$

$$0 + 2A - 3(At+B) = 5-5t$$
$$-3At + (2A-3B) = 5-5t$$

$-3A = -5$ $\qquad$ $2A - 3B = 5$

$A = 5/3$ $\qquad$ $2(5/3) - 3B = 5$

$$-3B = \frac{15}{3} - \frac{10}{3}$$
$$B = -5/9$$

$$y_p = \frac{5}{3}t - \frac{5}{9}$$

$$\boxed{y = y_c + y_p = C_1e^{-3t} + C_2e^{t} + \frac{5}{3}t - \frac{5}{9}}$$

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#7 continued

Eliminate y:
$$-D(5D-5)x - D(D-1)y = 0$$
$$(D-1)(4D-3)x + D(D-1)y = (D-1)t$$

$$(-5D^2+5D)x + (4D^2-7D+3)x = (D-1)t$$

$$(-D^2-2D+3)x = 1-t$$

This is a 2nd order linear eqn:

$$x'' + 2x' - 3x = t - 1$$

The corresp. homog. eqn.

$$x'' + 2x' - 3x = 0$$

has charact. eqn.

$$m^2 + 2m - 3 = 0$$
$$(m+3)(m-1) = 0 \longrightarrow x_c = c_1 e^{-3t} + c_2 e^t$$

Now we want a particular soln. of

$$x'' + 2x' - 3x = t - 1$$

UC set $S_1 = \{t, 1\}$

$$x_p = At + B$$
$$x_p' = A$$
$$x_p'' = 0$$

$$0 + 2A - 3(At+B) = t - 1$$

$-3At = t$ $\quad$ $2A - 3B = -1$

$A = -1/3$ $\quad$ $2(-1/3) - 3B = -1$

$B = 1/9$

$$x_p = -\tfrac{1}{3}t + \tfrac{1}{9}$$

$$x = x_c + x_p = \boxed{c_1 e^{-3t} + c_2 e^t - \tfrac{1}{3}t + \tfrac{1}{9}}$$

please note:
we could relate c_1, c_2, k_1, k_2 and eliminate two of them...

p.297 #9

Standard Differential Operator Method:

$$x' + y' - x - 6y = e^{3t} \longrightarrow (D-1)x + (D-6)y = e^{3t}$$

$$x' + 2y' - 2x - 6y = t \longrightarrow (D-2)x + (2D-6)y = t$$

Eliminate x: $(D-2)(D-1)x + (D-2)(D-6)y = (D-2)e^{3t}$

$$-(D-1)(D-2)x - (D-1)(2D-6)y = -(D-1)t$$

$$(D^2 - 8D + 12)y + (-2D^2 + 8D - 6)y = 3e^{3t} - 2e^{3t} - 1 + t$$

$$(-D^2 + 6)y = e^{3t} + t - 1$$

This is a 2nd order linear eqn:

$$y'' - 6y = -e^{3t} - t + 1$$

The corresp. homogen. eqn. $y'' - 6y = 0$ has charact. eqn.

$$m^2 - 6 = 0$$

$$m = \pm\sqrt{6} \rightarrow y_c = k_1 e^{\sqrt{6}t} + k_2 e^{-\sqrt{6}t}$$

Now we want a particular soln. of

$$y'' - 6y = -e^{3t} - t + 1$$

UC sets $S_1 = \{e^{3t}\}$

$S_2 = \{t, 1\}$

$$y_p = Ae^{3t} + Bt + C$$

$$y_p' = 3Ae^{3t} + B$$

$$y_p'' = 9Ae^{3t}$$

$$9Ae^{3t} - 6(Ae^{3t} + Bt + C) = -e^{3t} - t + 1$$

$3Ae^{3t} = -e^{3t}$ $-6Bt = -t$ $-6C = 1$

$A = -1/3$ $B = 1/6$ $C = -1/6$

$$y_p = -\frac{1}{3}e^{3t} + \frac{1}{6}t - \frac{1}{6}$$

$$\boxed{y = y_c + y_p = k_1 e^{\sqrt{6}t} + k_2 e^{-\sqrt{6}t} - \frac{1}{3}e^{3t} + \frac{1}{6}t - \frac{1}{6}}$$

p.297 #9 continued

Eliminate y:

$$-(2D-6)(D-1)x - (2D-6)(D-6)y = -(2D-6)e^{3t}$$
$$(D-6)(D-2)x + (2D-6)(D-6)y = (D-6)t$$

$$(-2D^2+8D-6)x + (D^2-8D+12)x = -6e^{3t}+6e^{3t}+1-6t$$

$$(-D^2+6)x = 1-6t$$

This is a 2nd order linear eqn:

$$x'' - 6x = 6t-1$$

The corresp. homog. eqn.

$x'' - 6x = 0$ has charact. eqn.

$$m^2 - 6 = 0$$
$$m = \pm\sqrt{6} \rightarrow x_c = c_1 e^{\sqrt{6}t} + c_2 e^{-\sqrt{6}t}$$

Now we want a particular soln. of

$$x'' - 6x = 6t-1$$

UC set $S_1 = \{t, 1\}$

$x_p = At + B$

$x_p' = A$

$x_p'' = 0$

$$0 - 6(At+B) = 6t-1$$

$-6At = 6t$ $\qquad$ $-6B = -1$

$A = -1$ $\qquad$ $B = 1/6$

$x_p = -t + 1/6$

$$x = x_c + x_p = c_1 e^{\sqrt{6}t} + c_2 e^{-\sqrt{6}t} - t + 1/6$$

Please Note: we could relate $c_1, c_2, k_1, + k_2$ to eliminate two of them...

p. 296 #1

Alternative Operator Method:

We still begin the problem the same as the standard operator method, solving first for x:

$$x = Ce^{-2t}$$

Now we want to eliminate Dy in the system:

$$\begin{aligned}(D-2)x + (D-4)y &= e^{t} \\ Dx + (D-1)y &= e^{4t}\end{aligned} \longrightarrow$$ subtract 2nd from 1st to get:

$$-2x - 3y = e^{t} - e^{4t}$$

substituting the known function x:

$$-2Ce^{-2t} - 3y = e^{t} - e^{4t}$$

$$\boxed{y = -\tfrac{1}{3}e^{t} + \tfrac{1}{3}e^{4t} - \tfrac{2}{3}Ce^{-2t}}$$

#3 p. 296

Alternative operator method:

Having solved for x using the standard method:

$$x = c_1e^{t} + c_2e^{3t} + t + 7/3$$

Now we want to eliminate Dy in the system:

$$\begin{aligned}(5D-3)x + (D+1)y &= 0 \\ (4D-3)x + Dy &= -3t\end{aligned} \longrightarrow$$ subtract 2nd from 1st to get:

$$Dx + y = 3t$$

substituting the known function x:

$$D(c_1e^{t} + c_2e^{3t} + t + 7/3) + y = 3t$$

$$c_1e^{t} + 3c_2e^{3t} + 1 + y = 3t$$

$$\boxed{y = -c_1e^{t} - 3c_2e^{3t} + 3t - 1}$$

p. 297 #5

Alternative Operator Method:

Having solved for x using the standard method:

$$x = \frac{1}{4}e^{t} + Ce^{-3t}$$

Now we want to eliminate Dy in the system:

$$\left.\begin{array}{l}(D-1)x + (D-3)y = e^{t} \\ (D+1)x + Dy = e^{3t}\end{array}\right\} \rightarrow$$ subtract 2nd from 1st:

$$-2x - 3y = e^{t} - e^{3t}$$

Substituting the known function x:

$$-2\left(\frac{1}{4}e^{t} + Ce^{-3t}\right) - 3y = e^{t} - e^{3t}$$

$$y = \frac{e^{t} - e^{3t} + \frac{1}{2}e^{t} + 2Ce^{-3t}}{-3}$$

$$y = \frac{1}{3}e^{3t} - \frac{1}{2}e^{t} - \frac{2}{3}Ce^{-3t}$$

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p. 297 #7

Alternative operator method:

Having solved for x using the standard method:

$$x = c_1e^{-3t} + c_2e^{t} - \frac{1}{3}t + \frac{1}{9}$$

Now we want to eliminate Dy in the system:

$$\left.\begin{array}{l}(5D-5)x + (D-1)y = 0 \\ (4D-3)x + Dy = t\end{array}\right\} \rightarrow$$ subtract 2nd from 1st to get:

$$(D-2)x - y = -t$$

substituting the known function x:

$$(D-2)\left(c_1e^{-3t} + c_2e^{t} - \frac{1}{3}t + \frac{1}{9}\right) - y = -t$$

$$-3c_1e^{-3t} - 2c_1e^{-3t} + c_2e^{t} - 2c_2e^{t} - \frac{1}{3} + \frac{2}{3}t - \frac{2}{9} - y = -t$$

$$y = -5c_1e^{-3t} - c_2e^{t} + \frac{5}{3}t - \frac{5}{9}$$

p. 296
#9

Alternate operator method:

Having solved for x using the standard method:

$$x = c_1 e^{\sqrt{6}t} + c_2 e^{-\sqrt{6}t} - t + 1/6$$

Now we want to eliminate Dy in the system:

$$(D-1)x + (D-6)y = e^{3t} \xrightarrow{*(-2)} (-2D+2)x + (-2D+12)y = -2e^{3t}$$

$$(D-2)x + (2D-6)y = t \longrightarrow (D-2)x + (2D-6)y = t$$

$$-Dx + 6y = -2e^{3t} + t$$

substituting the known function x:

$$-D(c_1 e^{\sqrt{6}t} + c_2 e^{-\sqrt{6}t} - t + 1/6) + 6y = -2e^{3t} + t$$

$$-\sqrt{6}c_1 e^{\sqrt{6}t} + \sqrt{6}c_2 e^{-\sqrt{6}t} + 1 + 6y = -2e^{3t} + t$$

$$\boxed{y = \frac{\sqrt{6}}{6}c_1 e^{\sqrt{6}t} - \frac{\sqrt{6}}{6}c_2 e^{-\sqrt{6}t} - \frac{1}{3}e^{3t} + \frac{1}{6}t - \frac{1}{6}}$$